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VECTORS (Q 2, PAPER 2)

LESSON NO. 3: DOT PRODUCT

2006

2 (b) $\vec{p} = -5\vec{i} + 2\vec{j}$, $\vec{q} = \vec{i} - 6\vec{j}$ and $\vec{r} = -\vec{i} + 5\vec{j}$.

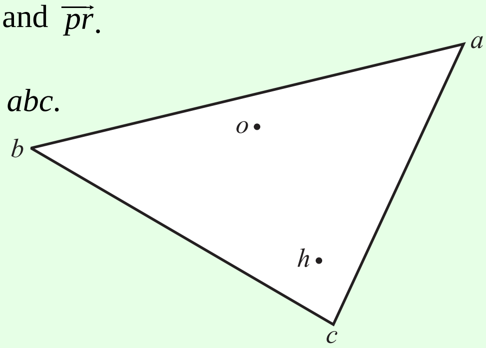
(i) Express $\overrightarrow{pq}$ and $\overrightarrow{pr}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Given that $10\vec{s} = |\overrightarrow{pr}|\,\overrightarrow{pq} + |\overrightarrow{pq}|\,\overrightarrow{pr}$, express $\vec{s}$ in terms of $\vec{i}$ and $\vec{j}$.

(iii) Find the measure of the angle between $\vec{s}$ and $\overrightarrow{pr}$.

2 (c) The origin o is the circumcentre of the triangle abc.

If $\vec{h} = \vec{a} + \vec{b} + \vec{c}$, show that $\overrightarrow{ah} \perp \overrightarrow{bc}$.



SOLUTION

2 (b) (i)

$\overrightarrow{ab} = \vec{b} - \vec{a}$ 1

$\overrightarrow{pq} = \vec{q} - \vec{p} = (\vec{i} - 6\vec{j}) - (-5\vec{i} + 2\vec{j}) = 6\vec{i} - 8\vec{j}$

$\overrightarrow{pr} = \vec{r} - \vec{p} = (-\vec{i} + 5\vec{j}) - (-5\vec{i} + 2\vec{j}) = 4\vec{i} + 3\vec{j}$

2(b) (ii)

$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow |\vec{r}| = \sqrt{x^2 + y^2}$ 5

$|\overrightarrow{pr}| = |4\vec{i} + 3\vec{j}| = \sqrt{16+9} = \sqrt{25} = 5$

$|\overrightarrow{pq}| = |6\vec{i} - 8\vec{j}| = \sqrt{36+64} = \sqrt{100} = 10$

$\therefore 10\vec{s} = |\overrightarrow{pr}|\overrightarrow{pq} + |\overrightarrow{pq}|\overrightarrow{pr} \Rightarrow 10\vec{s} = 5(6\vec{i} - 8\vec{j}) + 10(4\vec{i} + 3\vec{j})$

$\Rightarrow 10\vec{s} = 30\vec{i} - 40\vec{j} + 40\vec{i} + 30\vec{j} \Rightarrow 10\vec{s} = 70\vec{i} - 10\vec{j}$

$\Rightarrow \vec{s} = 7\vec{i} - \vec{j}$

2 (b) (iii)

$\overrightarrow{ab}.\overrightarrow{ac} = |\overrightarrow{ab}||\overrightarrow{ac}|\cos\theta$ 8

$\vec{i}.\vec{j} = \vec{j}.\vec{i} = 0$ 9

$\vec{i}.\vec{i} = \vec{j}.\vec{j} = |\vec{i}|^2 = |\vec{j}|^2 = 1$ 10

TRICK: Just multiply the $\vec{i}$ coefficients and multiply the $\vec{j}$ coefficients.

$\vec{s}.\overrightarrow{pr} = |\vec{s}||\overrightarrow{pr}|\cos\theta$

$\Rightarrow (7\vec{i} - \vec{j}).(4\vec{i} + 3\vec{j}) = |7\vec{i} - \vec{j}||4\vec{i} + 3\vec{j}|\cos\theta$

$\Rightarrow 25 = \sqrt{50}\sqrt{25}\cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ$

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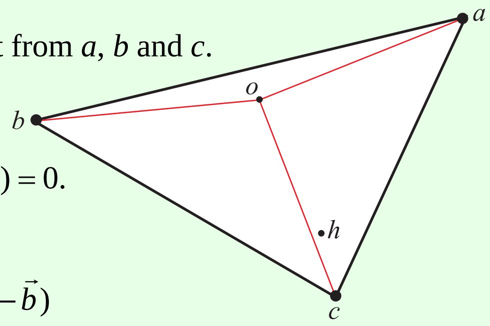
2 (c)

Dot Product Properties

1. $\vec{ab}.\vec{ac} = \vec{ac}.\vec{ab}$ [You can switch the order of dot product]
2. $\theta = 90^\circ \Leftrightarrow \vec{ab}.\vec{ac} = |\vec{ab}||\vec{ac}|\cos 90^\circ = 0$
   [If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]
3. $\vec{ab} = \vec{ac} \Leftrightarrow \vec{ab}.\vec{ab} = |\vec{ab}||\vec{ab}|\cos 0^\circ = |\vec{ab}|^2$
   [If you dot product a vector with itself you get the modulus squared. The modulus squared of a vector is got by dotting it with itself.]



If o is the circumcentre of triangle abc, o is equidistant from a, b and c.

$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}|$

To prove that $\vec{ah} \perp \vec{bc}$, you need to show that $(\vec{ah}).(\vec{bc}) = 0$.

$(\vec{ah}).(\vec{bc}) = (\vec{h} - \vec{a}).(\vec{c} - \vec{b})$

But $\vec{h} = \vec{a} + \vec{b} + \vec{c} \Rightarrow (\vec{a} + \vec{b} + \vec{c} - \vec{a}).(\vec{c} - \vec{b}) = (\vec{b} + \vec{c}).(\vec{c} - \vec{b})$

$= \vec{b}.\vec{c} - \vec{b}.\vec{b} + \vec{c}.\vec{c} - \vec{c}.\vec{b} = |\vec{c}|^2 - |\vec{b}|^2 = 0$

2005

2 (c) $\vec{u} = \vec{i} + 5\vec{j}$ and $\vec{v} = 4\vec{i} + 4\vec{j}$.

(i) Find $\cos \angle uov$, where o is the origin.

(ii) $\vec{r} = (1-k)\vec{u} + k\vec{v}$, where $k \in \mathbf{R}$ and $k \neq 0$. Find the value of k for which $|\angle uov| = |\angle vor|$.

Solution

2 (c) (i)

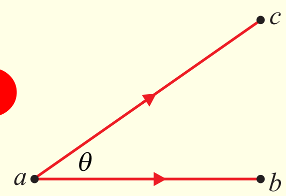
$\vec{ou} = \vec{i} + 5\vec{j}$, $\vec{ov} = 4\vec{i} + 4\vec{j}$

$\vec{ou}.\vec{ov} = |\vec{ou}||\vec{ov}|\cos \angle uov$

$\Rightarrow (\vec{i} + 5\vec{j}).(4\vec{i} + 4\vec{j}) = |\vec{i} + 5\vec{j}||4\vec{i} + 4\vec{j}|\cos \angle uov$

$\Rightarrow 24 = \sqrt{26}\sqrt{32}\cos \angle uov \Rightarrow \cos \angle uov = \dfrac{24}{4\sqrt{26}\sqrt{2}} = \dfrac{3}{\sqrt{13}}$

$\vec{ab}.\vec{ac} = |\vec{ab}||\vec{ac}|\cos\theta$ 8



$\vec{i}.\vec{j} = \vec{j}.\vec{i} = 0$ 9

Trick: Just multiply the $\vec{i}$ coefficients and multiply the $\vec{j}$ coefficients.

$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow |\vec{r}| = \sqrt{x^2 + y^2}$ 5

2 (c) (ii)

$\vec{or} = (1-k)\vec{u} + k\vec{v}$

$\Rightarrow \vec{or} = (1-k)(\vec{i} + 5\vec{j}) + k(4\vec{i} + 4\vec{j}) = (3k+1)\vec{i} + (5-k)\vec{j}$

If $|\angle uov| = |\angle vor| \Rightarrow \cos|\angle uov| = \cos|\angle vor| \Rightarrow \cos|\angle vor| = \frac{3}{\sqrt{13}}$

$\vec{or}.\vec{ov} = |\vec{or}||\vec{ov}|\cos\angle vor \Rightarrow \cos\angle vor = \dfrac{\vec{or}.\vec{ov}}{|\vec{or}||\vec{ov}|}$

$\Rightarrow \dfrac{3}{\sqrt{13}} = \dfrac{[(3k+1)\vec{i} + (5-k)\vec{j}].[4\vec{i} + 4\vec{j}]}{\left|(3k+1)\vec{i} + (5-k)\vec{j}\right|\left|4\vec{i} + 4\vec{j}\right|} \Rightarrow \dfrac{3}{\sqrt{13}} = \dfrac{4(3k+1) + 4(5-k)}{\sqrt{(3k+1)^2 + (5-k)^2}\sqrt{32}}$

$\Rightarrow \dfrac{3}{\sqrt{13}} = \dfrac{12k + 4 + 20 - 4k}{4\sqrt{(3k+1)^2 + (5-k)^2}\sqrt{2}} \Rightarrow \dfrac{3}{\sqrt{13}} = \dfrac{8k+24}{4\sqrt{(3k+1)^2 + (5-k)^2}\sqrt{2}}$

$\Rightarrow \dfrac{3}{\sqrt{13}} = \dfrac{2k+6}{\sqrt{(3k+1)^2 + (5-k)^2}\sqrt{2}} \Rightarrow 3\sqrt{(3k+1)^2 + (5-k)^2}\sqrt{2} = (2k+6)\sqrt{13}$

$\Rightarrow 18[(3k+1)^2 + (5-k)^2] = 13(2k+6)^2$

$\Rightarrow 18[10k^2 - 4k + 26] = 13(4k^2 + 24k + 36)$

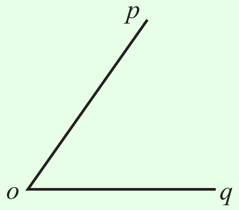
$\Rightarrow 180k^2 - 72k + 468 = 52k^2 + 312k + 468$

$\Rightarrow 128k^2 - 384k = 0 \Rightarrow k^2 - 3k = 0 \Rightarrow k(k-3) = 0$

$\Rightarrow k = 3.$ (You were told that $k \neq 0.$)

2004

2 (c) p and q are points and o is the origin. p, q and o are not collinear and $|\vec{p}| = |\vec{q}|$.



(i) Prove that $\overrightarrow{pq}$ is perpendicular to $(\vec{p}+\vec{q})$.

(ii) Prove that $\overrightarrow{po}.\overrightarrow{pq} = \frac{1}{2}|\overrightarrow{pq}|^2$.

SOLUTION

2 (c) (i)

DOT PRODUCT PROPERTIES

1. $\overrightarrow{ab}.\overrightarrow{ac} = \overrightarrow{ac}.\overrightarrow{ab}$ [You can switch the order of dot product]

2. $\theta = 90^\circ \Leftrightarrow \overrightarrow{ab}.\overrightarrow{ac} = |\overrightarrow{ab}||\overrightarrow{ac}|\cos 90^\circ = 0$
[If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]

3. $\overrightarrow{ab} = \overrightarrow{ac} \Leftrightarrow \overrightarrow{ab}.\overrightarrow{ab} = |\overrightarrow{ab}||\overrightarrow{ab}|\cos 0^\circ = |\overrightarrow{ab}|^2$
[If you dot product a vector with itself you get the modulus squared. The modulus squared of a vector is got by dotting it with itself.]

$\overrightarrow{pq}.(\vec{p}+\vec{q}) = (\vec{q}-\vec{p}).(\vec{p}+\vec{q}) = \vec{q}.\vec{p}+\vec{q}.\vec{q}-\vec{p}.\vec{p}-\vec{p}.\vec{q}$

$= |\vec{q}|^2 - |\vec{p}|^2 = 0$ as you are told that $|\vec{p}| = |\vec{q}|$.

Therefore, $\overrightarrow{pq}$ is perpendicular to $(\vec{p}+\vec{q})$ as their dot product is zero.

2 (c) (ii)

$\overrightarrow{po}.\overrightarrow{pq} = -\vec{p}.(\vec{q}-\vec{p}) = \vec{p}.\vec{p}-\vec{p}.\vec{q} = |\vec{p}|^2 - \vec{p}.\vec{q}$ (LHS)

$\frac{1}{2}|\overrightarrow{pq}|^2 = \frac{1}{2}\overrightarrow{pq}.\overrightarrow{pq} = \frac{1}{2}(\vec{q}-\vec{p}).(\vec{q}-\vec{p}) = \frac{1}{2}(\vec{q}.\vec{q}-2\vec{p}.\vec{q}+\vec{p}.\vec{p})$

$= \frac{1}{2}(|\vec{q}|^2 - 2\vec{p}.\vec{q} + |\vec{p}|^2) = \frac{1}{2}(-2\vec{p}.\vec{q}+2|\vec{p}|^2) = -\vec{p}.\vec{q}+|\vec{p}|^2$ (RHS)

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2003

2 (b) $\vec{p}=2\vec{i}+\vec{j},\ \vec{q}=3\vec{i}+k\vec{j},\ \vec{r}=3\vec{i}+t\vec{j}$ where $k,\ t\in\mathbf{R}$ and o is the origin.

(i) Given that $\vec{p}\perp\vec{q}$, calculate the value of k.

(ii) Given that $|\angle por|=45^\circ$, calculate the two possible values of t.

SOLUTION

2 (b) (i)

> **2.** $\theta=90^\circ \Leftrightarrow \overrightarrow{ab}.\overrightarrow{ac}=|\overrightarrow{ab}||\overrightarrow{ac}|\cos 90^\circ=0$
> [If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]

If $\vec{p}\perp\vec{q}\Rightarrow \vec{p}.\vec{q}=0\Rightarrow(2\vec{i}+\vec{j}).(3\vec{i}+k\vec{j})=0$

$\Rightarrow 6+k=0\Rightarrow k=-6$

> **TRICK**: Just multiply the $\vec{i}$ coefficients and multiply the $\vec{j}$ coefficients.

2 (b) (ii)

> $\overrightarrow{ab}.\overrightarrow{ac}=|\overrightarrow{ab}||\overrightarrow{ac}|\cos\theta$ **8**

$\overrightarrow{op}.\overrightarrow{or}=|\overrightarrow{op}||\overrightarrow{or}|\cos 45^\circ$

$\Rightarrow(2\vec{i}+\vec{j}).(3\vec{i}+t\vec{j})=|2\vec{i}+\vec{j}||3\vec{i}+t\vec{j}|(\tfrac{1}{\sqrt{2}})$

> $\vec{r}=x\vec{i}+y\vec{j}\Rightarrow|\vec{r}|=\sqrt{x^2+y^2}$ **5**

$\Rightarrow 6+t=\dfrac{\sqrt{5}\sqrt{9+t^2}}{\sqrt{2}}\Rightarrow\sqrt{2}(6+t)=\sqrt{5}\sqrt{9+t^2}$

$\Rightarrow 2(6+t)^2=5(9+t^2)\Rightarrow 2(36+12t+t^2)=45+5t^2$

$\Rightarrow 72+24t+2t^2=45+5t^2\Rightarrow 3t^2-24t-27=0$

$\Rightarrow t^2-8t-9=0\Rightarrow(t-9)(t+1)=0\Rightarrow t=-1,\ 9$

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2002

2 (c) $\vec{k} = \vec{i} + 3\vec{j}$, $\vec{n} = 4\vec{i} - 2\vec{j}$, $\vec{u} = 2\vec{i} + \vec{j}$ and $\vec{v} = x\vec{i} + y\vec{j}$ where $x, y \in \mathbf{R}$.

(i) Express the value of $\vec{kn}.\vec{kv}$ in the form $ax + by + c$ where $a, b, c \in \mathbf{R}$.

(ii) Prove that if $\vec{kn}.\vec{kv} = \vec{kn}.\vec{ku}$, and $\vec{u} \neq \vec{v}$, then $\vec{kn} \perp \vec{uv}$.

SOLUTION

2 (c) (i)

$\vec{kn}.\vec{kv} = (\vec{n} - \vec{k}).(\vec{v} - \vec{k}) = (4\vec{i} - 2\vec{j} - \vec{i} - 3\vec{j}).(x\vec{i} + y\vec{j} - \vec{i} - 3\vec{j})$

$= (3\vec{i} - 5\vec{j}).((x-1)\vec{i} + (y-3)\vec{j}) = 3(x-1) - 5(y-3)$

$= 3x - 3 - 5y + 15 = 3x - 5y + 12$

2 (c) (ii)

If $\vec{kn}.\vec{kv} = \vec{kn}.\vec{ku} \Rightarrow 3x - 5y + 12 = (\vec{n} - \vec{k}).(\vec{u} - \vec{k})$

$\Rightarrow 3x - 5y + 12 = (4\vec{i} - 2\vec{j} - \vec{i} - 3\vec{j}).(2\vec{i} + \vec{j} - \vec{i} - 3\vec{j})$

$\Rightarrow 3x - 5y + 12 = (3\vec{i} - 5\vec{j}).(\vec{i} - 2\vec{j}) = 13 \Rightarrow 3x - 5y - 1 = 0$ ....**(1)**

TRICK: Just multiply the $\vec{i}$ coefficients and multiply the $\vec{j}$ coefficients.

2. $\theta = 90^\circ \Leftrightarrow \vec{ab}.\vec{ac} = |\vec{ab}||\vec{ac}|\cos 90^\circ = 0$

[If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]

You can prove that $\vec{kn} \perp \vec{uv}$ by showing that their dot product is zero.

$\vec{kn}.\vec{uv} = (\vec{n} - \vec{k}).(\vec{v} - \vec{u}) = (4\vec{i} - 2\vec{j} - \vec{i} - 3\vec{j}).(x\vec{i} + y\vec{j} - 2\vec{i} - \vec{j})$

$= (3\vec{i} - 5\vec{j}).((x-2)\vec{i} + (y-1)\vec{j}) = 3(x-2) - 5(y-1)$

$= 3x - 6 - 5y + 5 = 3x - 5y - 1 = 0$ (From equation (**1**)).

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2001

2 (c) *rst* is a triangle where $\vec{r} = -\vec{i} + 2\vec{j}$, $\vec{s} = -4\vec{i} - 2\vec{j}$ and $\vec{t} = 3\vec{i} - \vec{j}$.

(i) Express $\vec{rs}$, $\vec{st}$ and $\vec{tr}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Show that the triangle *rst* is right-angled at *r*.

(iii) Find the measure of $\angle rst$.

SOLUTION

2 (c) (i)

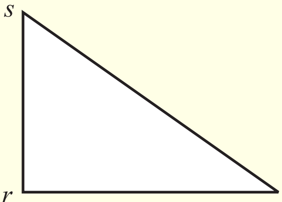
$\vec{rs} = \vec{s} - \vec{r} = -4\vec{i} - 2\vec{j} + \vec{i} - 2\vec{j} = -3\vec{i} - 4\vec{j}$

$\vec{st} = \vec{t} - \vec{s} = 3\vec{i} - \vec{j} + 4\vec{i} + 2\vec{j} = 7\vec{i} + \vec{j}$ $\qquad \boxed{\vec{ab} = \vec{b} - \vec{a}}$ 1

$\vec{tr} = \vec{r} - \vec{t} = -\vec{i} + 2\vec{j} - 3\vec{i} + \vec{j} = -4\vec{i} + 3\vec{j}$

2 (c) (ii)

2. $\theta = 90^\circ \Leftrightarrow \vec{ab}.\vec{ac} = |\vec{ab}||\vec{ac}|\cos 90^\circ = 0$
[If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]



TRICK: Just multiply the $\vec{i}$ coefficients and multiply the $\vec{j}$ coefficients when doing the dot product..

You need to show that $\vec{sr}.\vec{rt} = 0$.

$\vec{rs}.\vec{tr} = (-3\vec{i} - 4\vec{j}).(-4\vec{i} + 3\vec{j}) = 12 - 12 = 0$

2 (c) (iii)

$\boxed{\vec{ab}.\vec{ac} = |\vec{ab}||\vec{ac}|\cos\theta}$ 8

$$\vec{sr}.\vec{st} = |\vec{sr}||\vec{st}|\cos\angle rst \Rightarrow \cos\angle rst = \frac{\vec{sr}.\vec{st}}{|\vec{sr}||\vec{st}|}$$

$$\Rightarrow \cos\angle rst = \frac{(3\vec{i} + 4\vec{j}).(7\vec{i} + \vec{j})}{|3\vec{i} + 4\vec{j}||7\vec{i} + \vec{j}|} \Rightarrow \cos\angle rst = \frac{21 + 4}{\sqrt{25}\sqrt{50}} = \frac{25}{5 \times 5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \angle rst = 45^\circ$$