VECTORS (Q 2, PAPER 2)

2011

2. **(a)** Find the value of s and the value of t that satisfy the equation

$$s(\vec{i}-4\vec{j})+t(2\vec{i}+3\vec{j})=4\vec{i}-27\vec{j}.$$

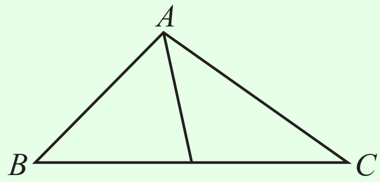
(b) $\overrightarrow{OP}=3\vec{i}-4\vec{j}$ and $\overrightarrow{OQ}=5(\overrightarrow{OP}^{\perp})$, where O is the origin.

(i) Find $\overrightarrow{OQ}$ in terms of $\vec{i}$ and $\vec{j}$.

.

(ii) Find $\cos|\angle OQP|$, in surd form.

(c) ABC is a triangle and D is the mid-point of $[BC]$.



(i) Express $\overrightarrow{AB}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$ and express $\overrightarrow{AC}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

(ii) Hence, prove that $|AB|^2+|AC|^2=2|AD|^2+\frac{1}{2}|BC|^2$.

SOLUTION

2 (a)

> **ADDITION AND MULTIPLICATION BY A SCALAR (NUMBER)**
> Multiply out the brackets and add the $\vec{i}$'s together and the $\vec{j}$'s together.
>
> **EQUALITY**
> If $a\vec{i}+b\vec{j}=c\vec{i}+d\vec{j}$ then $a=c$ and $b=d$. Equate the $\vec{i}$'s and the $\vec{j}$'s.

$$s(\vec{i}-4\vec{j})+t(2\vec{i}+3\vec{j})=4\vec{i}-27\vec{j}$$
$$s\vec{i}-4s\vec{j}+2t\vec{i}+3t\vec{j}=4\vec{i}-27\vec{j}$$
$$s\vec{i}+2t\vec{i}-4s\vec{j}+3t\vec{j}=4\vec{i}-27\vec{j}$$
$$(s+2t)\vec{i}+(-4s+3t)\vec{j}=4\vec{i}-27\vec{j}$$

$$s+2t=4.......(\mathbf{1})(\times 4)$$
$$-4s+3t=-27...(\mathbf{2})$$

$$4s+8t=16.......(\mathbf{1})(\times 4)$$
$$-4s+3t=-27....(\mathbf{2})$$
$$11t=-11\Rightarrow t=-1$$

Substitute this value of t into Eqn. (**1**): $s+2(-1)=4$

$$s-2=4$$
$$\therefore s=6$$

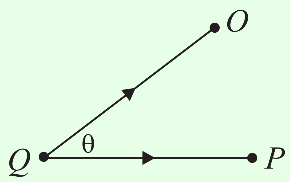
2 (b) (i)

$\overrightarrow{OP} = 3\vec{i} - 4\vec{j}$

$\overrightarrow{OP}^{\perp} = 4\vec{i} + 3\vec{j}$ $\quad$ $\vec{r} = +x\vec{i} + y\vec{j} \Rightarrow \vec{r}^{\perp} = -y\vec{i} + x\vec{j}$

$\overrightarrow{OQ} = 5(\overrightarrow{OP}^{\perp}) = 5(4\vec{i} + 3\vec{j}) = 20\vec{i} + 15\vec{j}$

2 (b) (ii)



Dot product: $\overrightarrow{ab}.\overrightarrow{ac} = |\overrightarrow{ab}||\overrightarrow{ac}|\cos\theta$

$\overrightarrow{QO}.\overrightarrow{QP} = |\overrightarrow{QO}||\overrightarrow{QP}|\cos\theta$

$\overrightarrow{OQ} = -\overrightarrow{QO} = -20\vec{i} - 15\vec{j}$

$\overrightarrow{QP} = \vec{P} - \vec{Q} = (3\vec{i} - 4\vec{j}) - (20\vec{i} + 15\vec{j})$ $\quad$ $\overrightarrow{ab} = \vec{b} - \vec{a}$

$= -17\vec{i} - 19\vec{j}$

$\overrightarrow{QO}.\overrightarrow{QP} = |\overrightarrow{QO}||\overrightarrow{QP}|\cos\theta$

$(-20\vec{i} - 15\vec{j}).(-17\vec{i} - 19\vec{j}) = |-20\vec{i} - 15\vec{j}||-17\vec{i} - 19\vec{j}|\cos\theta$ $\quad$ $|\vec{r}| = \sqrt{x^2 + y^2}$

$(-20)(-17) + (-15)(-19) = \sqrt{(-20)^2 + (-15)^2}\sqrt{(-17)^2 + (-19)^2}\cos\theta$

$625 = \sqrt{625}\sqrt{650}\cos\theta$

$625 = 25 \times 5\sqrt{26}\cos\theta$

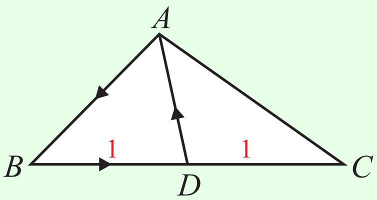
$\therefore \cos\theta = \dfrac{5}{\sqrt{26}}$

Dot Product of $\{\vec{i}.\vec{j}\}$: Just multiply the $\vec{i}$ coefficients and multiply the $\vec{j}$ coefficients.

2 (c) (i)

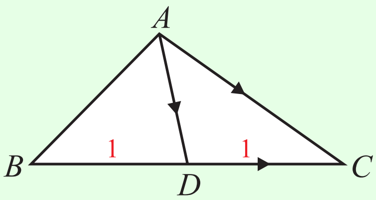
$\overrightarrow{AB} = \overrightarrow{BD} + \overrightarrow{DA}$

$= \tfrac{1}{2}\overrightarrow{BC} - \overrightarrow{AD}$



$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

$= \overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{BC}$



2 (c) (ii)

Dot Product Properties

$\overrightarrow{AB}.\overrightarrow{AC}=\overrightarrow{AC}.\overrightarrow{AB}$ [You can switch the order of dot product]

$\left|\overrightarrow{AB}\right|^2=\overrightarrow{AB}.\overrightarrow{AB}$ [If you dot product a vector with itself you get the modulus squared. The modulus squared of a vector is got by dotting it with itself.]

$$\begin{aligned}\left|\overrightarrow{AB}\right|^2=\overrightarrow{AB}.\overrightarrow{AB}&=(\tfrac{1}{2}\overrightarrow{BC}-\overrightarrow{AD}).(\tfrac{1}{2}\overrightarrow{BC}-\overrightarrow{AD})\\&=\tfrac{1}{4}\overrightarrow{BC}.\overrightarrow{BC}-\tfrac{1}{2}\overrightarrow{BC}.\overrightarrow{AD}-\tfrac{1}{2}\overrightarrow{AD}.\overrightarrow{BC}+\overrightarrow{AD}.\overrightarrow{AD}\\&=\tfrac{1}{4}\left|\overrightarrow{BC}\right|^2-\overrightarrow{BC}.\overrightarrow{AD}+\left|\overrightarrow{AD}\right|^2\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{AC}\right|^2=\overrightarrow{AC}.\overrightarrow{AC}&=(\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{BC}).(\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{BC})\\&=\overrightarrow{AD}.\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{AD}.\overrightarrow{BC}+\tfrac{1}{2}\overrightarrow{BC}.\overrightarrow{AD}+\tfrac{1}{4}\overrightarrow{BC}.\overrightarrow{BC}\\&=\left|\overrightarrow{AD}\right|^2+\overrightarrow{BC}.\overrightarrow{AD}+\tfrac{1}{4}\left|\overrightarrow{BC}\right|^2\end{aligned}$$

$$\left|\overrightarrow{AB}\right|^2=\tfrac{1}{4}\left|\overrightarrow{BC}\right|^2-\cancel{\overrightarrow{BC}.\overrightarrow{AD}}+\left|\overrightarrow{AD}\right|^2$$

$$\left|\overrightarrow{AC}\right|^2=\left|\overrightarrow{AD}\right|^2+\cancel{\overrightarrow{AD}.\overrightarrow{BC}}+\tfrac{1}{4}\left|\overrightarrow{BC}\right|^2$$

$$\therefore\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{AC}\right|^2=2\left|\overrightarrow{AD}\right|^2+\tfrac{1}{2}\left|\overrightarrow{BC}\right|^2$$