

## TRIGONOMETRY (Q 4 & 5, PAPER 2)

**1999**

- 4 (a) Find the value of  $k$  for which

$$\sin 75^\circ - \sin 15^\circ = \frac{1}{\sqrt{k}}, k \in \mathbf{N}_0.$$

See tables page 9.

- (b) (i) Express  $\sin 5x - \sin x$  as a product of sine and cosine.

- (ii) Find all the solutions of the equation

$$\sin 5x - \sin x = 0$$

in the domain  $0^\circ \leq x \leq 180^\circ$ .

- (c) Prove that

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Find, in the form  $p \pm \sqrt{q}$ ,  $p \in \mathbf{Z}$ ,  $q \in \mathbf{N}$ ,

(i)  $\tan 75^\circ$

(ii)  $\tan 15^\circ$ .

### SOLUTION

**4 (a)**

Change the sum into a product:

$$\sin 75^\circ - \sin 15^\circ = 2 \cos\left(\frac{75^\circ + 15^\circ}{2}\right) \sin\left(\frac{75^\circ - 15^\circ}{2}\right) = 2 \cos 45^\circ \sin 30^\circ$$

$$= 2 \left(\frac{1}{\sqrt{2}}\right) \left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \quad [\text{Using the table below.}]$$

$A$	0	$\pi$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$
$A$	$0^\circ$	$180^\circ$	$90^\circ$	$60^\circ$	$45^\circ$	$30^\circ$
$\cos A$	1	-1	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\sin A$	0	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan A$	0	0	$\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

$$\therefore k = 2$$

### SUMS $\rightarrow$ PRODUCTS

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

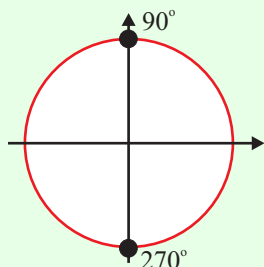
**4 (b) (i)**

$$\sin 5x - \sin x = 2 \cos\left(\frac{5x+x}{2}\right) \sin\left(\frac{5x-x}{2}\right) = 2 \cos 3x \sin 2x$$

**4 (b) (ii)**

Solve  $2 \cos 3x \sin 2x = 0 \Rightarrow \cos 3x \sin 2x = 0$

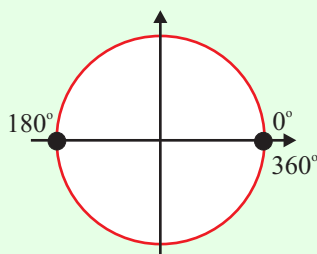
$\therefore \cos 3x = 0$



$$3x = 90^\circ, 450^\circ \\ = 270^\circ, 630^\circ$$

$$x = 30^\circ, 150^\circ \\ = 90^\circ, 210^\circ$$

$\therefore \sin 2x = 0$



$$2x = 0^\circ, 360^\circ \\ = 180^\circ, 540^\circ$$

$$x = 0^\circ, 180^\circ \\ = 90^\circ, 270^\circ$$

$\therefore x = 0^\circ, 30^\circ, 90^\circ, 150^\circ, 180^\circ$

SUMS $\rightarrow$ PRODUCTS
$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$
$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$
$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$
$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$

**4 (c)**

This is a trig identity.

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| <p><b>STEPS</b></p> <ol style="list-style-type: none"> <li>1. You have to prove that the left-hand side (<i>LHS</i>) equals the right-hand side (<i>RHS</i>).</li> <li>2. Change everything to sine and cosine.</li> <li>3. Simplify each side using page 9 of the tables and good algebra.</li> <li>4. For half angles, <math>\frac{\theta}{2}</math>: Let <math>\frac{\theta}{2} = A \Rightarrow \theta = 2A</math>.</li> </ol> |
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*LHS*

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

*RHS*

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)} \times \frac{\cos A \cos B}{\cos A \cos B}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$LHS = RHS$

**4 (c) (i)**

$\tan 75^\circ = \tan(45^\circ + 30^\circ)$  [Written as a sum of famous angles whose values are given in the tables on page 9.]

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1)(\frac{1}{\sqrt{3}})}$$

Multiply above and below by  $\sqrt{3}$ .

$$\therefore \tan 75^\circ = \frac{(1 + \frac{1}{\sqrt{3}}) \times \sqrt{3}}{(1 - \frac{1}{\sqrt{3}}) \times \sqrt{3}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

Rationalise the denominator by multiplying above and below by the conjugate.

$$\therefore \tan 75^\circ = \frac{(\sqrt{3} + 1) \times (\sqrt{3} + 1)}{(\sqrt{3} - 1) \times (\sqrt{3} + 1)} = \frac{3 + \sqrt{3} + \sqrt{3} + 1}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

A	0	$\pi$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$
A	$0^\circ$	$180^\circ$	$90^\circ$	$60^\circ$	$45^\circ$	$30^\circ$
cos A	1	-1	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
sin A	0	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan A	0	0	$\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

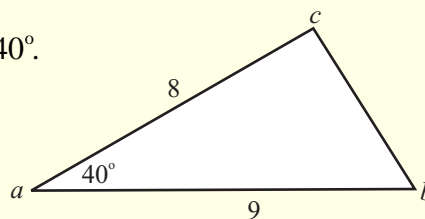
**4 (c) (ii)**

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$  [Change the sign in the original formula]

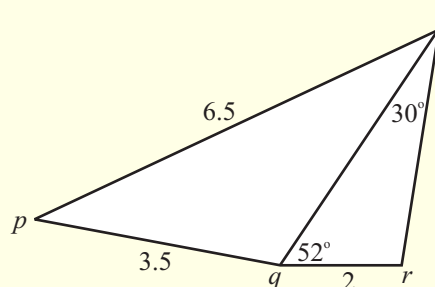
$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{(1 - \frac{1}{\sqrt{3}}) \times \sqrt{3}}{(1 + \frac{1}{\sqrt{3}}) \times \sqrt{3}}$$

$$= \frac{(\sqrt{3} - 1) \times (\sqrt{3} - 1)}{(\sqrt{3} + 1) \times (\sqrt{3} - 1)} = \frac{3 - 2\sqrt{3} + 1}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

- 5 (a) In the triangle  $abc$ ,  $|ab| = 9$ ,  $|ac| = 8$  and  $|\angle cab| = 40^\circ$ .  
Find the area of triangle  $abc$ , correct to two places of decimals.



- (b) In the triangles  $pqs$  and  $qrs$ ,  $|pq| = 3.5$ ,  $|qr| = 2$ ,  $|ps| = 6.5$ ,  $|\angle qsr| = 30^\circ$  and  $|\angle sqr| = 52^\circ$ .  
Calculate



- (i)  $|qs|$ , correct to two places of decimals  
(ii)  $|\angle pqs|$ , correct to the nearest degree.
- (c) Express  $\sin(135^\circ - A)$  in terms of  $\sin A$  and  $\cos A$ .

Express  $\sin(135^\circ - A) \cos(135^\circ + A)$  in the form  $k(1 + \sin pA)$ , where  $k, p \in \mathbf{R}$ .

Find the values of  $A$  for which

$$\sin(135^\circ - A) \cos(135^\circ + A) = -\frac{3}{4}$$

where  $0^\circ \leq A \leq 180^\circ$ .

### SOLUTION

#### 5 (a)

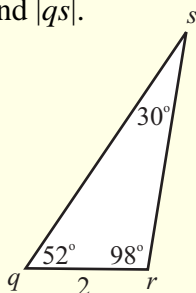
$$A = \frac{1}{2}(8)(9) \sin 40^\circ = 23.14 \text{ square units}$$

$$A = \frac{1}{2}ab \sin C \dots\dots \mathbf{5}$$

#### 5 (b) (i)

Start with triangle  $qrs$  as it has the most information. It is easy to find the third angle.

Use the Sine Rule to find  $|qs|$ .



#### THE SINE RULE

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \dots\dots \mathbf{18}$$

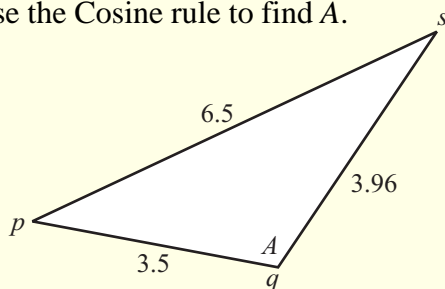
You use the Sine Rule when you are given:

1. 2 sides and 1 non-included angle,
2. 2 angles and 1 side.

$$\therefore \frac{|qs|}{\sin 98^\circ} = \frac{2}{\sin 30^\circ} \Rightarrow |qs| = \frac{2 \sin 98^\circ}{\sin 30^\circ} = 3.96 \text{ square units}$$

**5 (b) (ii)**

Move into triangle  $pqs$ . Call the angle you need to find,  $A$ . Use the Cosine rule to find  $A$ .



$$\therefore 6.5^2 = 3.5^2 + 3.96^2 - 2(3.5)(3.96) \cos A$$

$$\Rightarrow A = \cos^{-1} \left( \frac{6.5^2 - 3.5^2 + 3.96^2}{-2(3.5)(3.96)} \right) = 121^\circ$$

**THE COSINE RULE**

$$a^2 = b^2 + c^2 - 2bc \cos A \dots\dots \mathbf{19}$$

You use the Cosine rule when you are given:

1. 2 sides and 1 included angle,
2. 3 sides.

**5 (c)**

Use formula 10.

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \dots\dots \mathbf{10}$$

$$\sin(135^\circ - A) = \sin 135^\circ \cos A - \cos 135^\circ \sin A$$

[Using ASTC,  $\sin 135^\circ = \sin 45^\circ$  and  $\cos 135^\circ = -\cos 45^\circ$ ]

$$\therefore \sin(135^\circ - A) = \sin 45^\circ \cos A + \cos 45^\circ \sin A = \frac{1}{\sqrt{2}} \cos A + \frac{1}{\sqrt{2}} \sin A \text{ [Using the table on p.9]}$$

$$\therefore \sin(135^\circ - A) = \frac{1}{\sqrt{2}} (\cos A + \sin A)$$

$$\cos(135^\circ + A) = \cos 135^\circ \cos A - \sin 135^\circ \sin A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \dots\dots \mathbf{11}$$

$$= -\cos 45^\circ \cos A - \sin 45^\circ \sin A = -\frac{1}{\sqrt{2}} \cos A - \frac{1}{\sqrt{2}} \sin A = -\frac{1}{\sqrt{2}} (\cos A + \sin A)$$

$$\therefore \sin(135^\circ - A) \cos(135^\circ + A) = \left[ \frac{1}{\sqrt{2}} (\cos A + \sin A) \right] \left[ -\frac{1}{\sqrt{2}} (\cos A + \sin A) \right]$$

$$= -\frac{1}{2} (\cos A + \sin A)^2 = -\frac{1}{2} (\cos^2 A + 2 \sin A \cos A + \sin^2 A)$$

$$\cos^2 A + \sin^2 A = 1 \dots\dots \mathbf{8}$$

$$= -\frac{1}{2} (1 + \sin 2A) \text{ [Using formulae 8 and 13.]}$$

$$\sin 2A = 2 \sin A \cos A \dots\dots \mathbf{13}$$

$$\sin(135^\circ - A) \cos(135^\circ + A) = -\frac{3}{4} \Rightarrow -\frac{1}{2} (1 + \sin 2A) = -\frac{3}{4}$$

$$\Rightarrow 1 + \sin 2A = \frac{3}{2} \Rightarrow \sin 2A = \frac{1}{2}$$

$$\Rightarrow 2A = \sin^{-1} \left( \frac{1}{2} \right) = 30^\circ$$

$$2A = 30^\circ, 390^\circ \text{ (First quadrant)} \\ = 150^\circ \text{ (Second quadrant)}$$

$$A = 15^\circ, 195^\circ \\ = 75^\circ$$

$$\therefore A = 15^\circ, 75^\circ$$

