

TRIGONOMETRY (Q 4 & 5, PAPER 2)

2011

4. (a) Evaluate $\lim_{x \rightarrow \infty} \left(\frac{\sin 2x + \sin x}{3x} \right)$.

(b) Find all the solutions of the equation

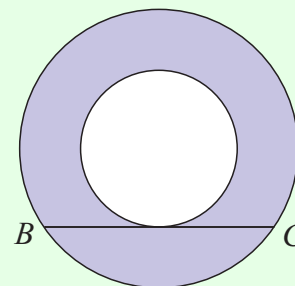
$$\sin 2x + \cos x = 0, \text{ where } 0^\circ \leq x \leq 360^\circ.$$

(c) The diagram shows two concentric circles.

A tangent to the inner circle cuts the outer circle at B and C , where $|BC| = 2x$.

(i) Express the area of the shaded region in terms of x .

(ii) In the case where the radius of the outer circle is $2x$, show that the portion of the shaded region that lies below BC has area $\left(\frac{2\pi}{3} - \sqrt{3} \right) x^2$.



SOLUTION

4 (a)

$$\begin{aligned} & \lim_{x \rightarrow 0} \left(\frac{\sin 2x + \sin x}{3x} \right) && \boxed{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{3x} \right) + \lim_{x \rightarrow 0} \left(\frac{\sin x}{3x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{2x} \times \frac{2x}{3x} \right) + \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \times \frac{x}{3x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{2x} \right) \times \frac{2}{3} + \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \times \frac{1}{3} \\ &= 1 \times \frac{2}{3} + 1 \times \frac{1}{3} \\ &= 1 \end{aligned}$$

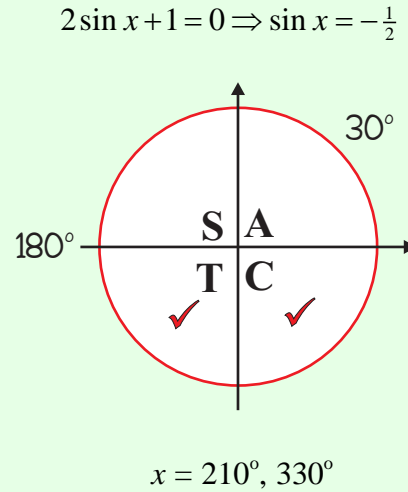
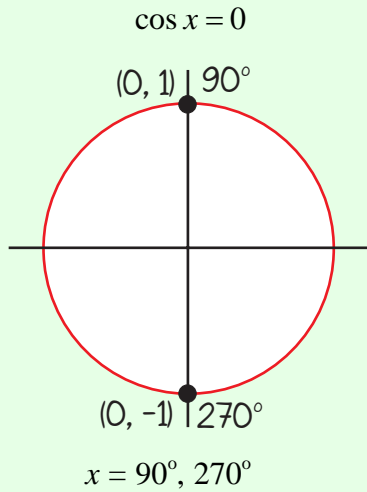
4 (b)

$$\sin 2x + \cos x = 0$$

$$2 \sin x \cos x + \cos x = 0$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos x(2 \sin x + 1) = 0$$



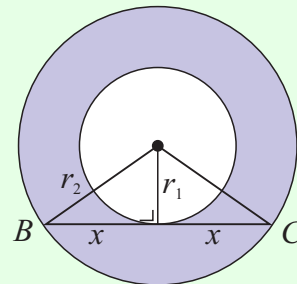
Answer: $x = 90^\circ, 210^\circ, 270^\circ, 330^\circ$

4 (c)

Call r_1 , the radius of the smaller circle and r_2 , the radius of the bigger circle.

$$\therefore r_2^2 = r_1^2 + x^2$$

$$r_2 = \sqrt{r_1^2 + x^2}$$



4 (c) (i)

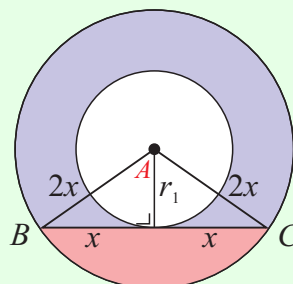
Area of shaded region = Area of bigger circle – Area of smaller circle

$$\begin{aligned} &= \pi r_2^2 - \pi r_1^2 \\ &= \pi (\sqrt{r_1^2 + x^2})^2 - \pi r_1^2 \\ &= \pi (r_1^2 + x^2) - \pi r_1^2 \\ &= \pi r_1^2 + \pi x^2 - \pi r_1^2 \\ &= \pi x^2 \end{aligned}$$

4 (c) (ii)

Calculate the angle A shown.

$$\sin A = \frac{x}{2x} = \frac{1}{2} \Rightarrow A = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$$



Call the centre of the circles O .

$$\therefore |\angle BOC| = 60^\circ$$

Area of portion below BC
 = Area of sector OBC – Area of triangle OBC

$$A = \frac{1}{2}r^2\theta$$

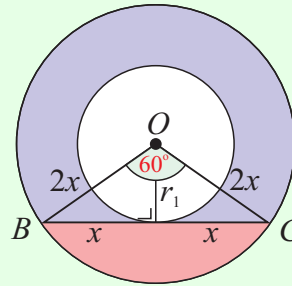
$$A = \frac{1}{2}ab \sin C$$

$$= \frac{1}{2}(2x)^2\left(\frac{\pi}{3}\right) - \frac{1}{2}(2x)(2x)\sin 60^\circ$$

$$= \frac{1}{2}(4x^2)\left(\frac{\pi}{3}\right) - 2x^2\left(\frac{\sqrt{3}}{2}\right)$$

$$= 2x^2\left(\frac{\pi}{3}\right) - \sqrt{3}x^2$$

$$= \left(\frac{2\pi}{3} - \sqrt{3}\right)x^2$$



5. (a) Find the values of x for which $3 \tan x = \sqrt{3}$, where $0^\circ \leq x \leq 360^\circ$.

(b) (i) Prove that $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

(ii) Show that if $\alpha + \beta = 90^\circ$, then $\frac{\tan 2\alpha}{\tan 2\beta} = -1$.

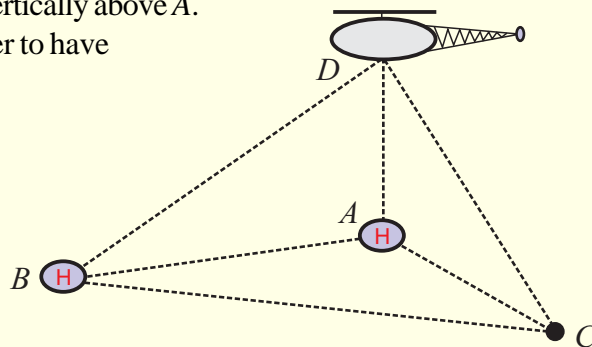
(c) A and B are two helicopter landing pads on level ground. C is another point on the same level ground. $|BC| = 800$ metres, $|AC| = 900$ metres, and $|\angle BCA| = 60^\circ$.

A helicopter at point D is hovering vertically above A .

A person at C observes the helicopter to have an angle of elevation of 30° .

(i) Find $|AD|$, in surd form.

(ii) Find $|BD|$.



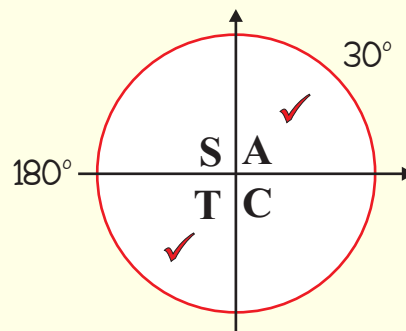
SOLUTION

5 (a)

$$3 \tan x = \sqrt{3}$$

$$\tan x = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \Rightarrow x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

$$x = 30^\circ, 210^\circ$$



5 (b) (i)

LHS

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}\end{aligned}$$

RHS

$$\begin{aligned}\frac{\tan A + \tan B}{1 - \tan A \tan B} &= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)} \\ &= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)} \times \frac{\cos A \cos B}{\cos A \cos B} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}\end{aligned}$$

$$LHS = RHS$$

5 (b) (ii)

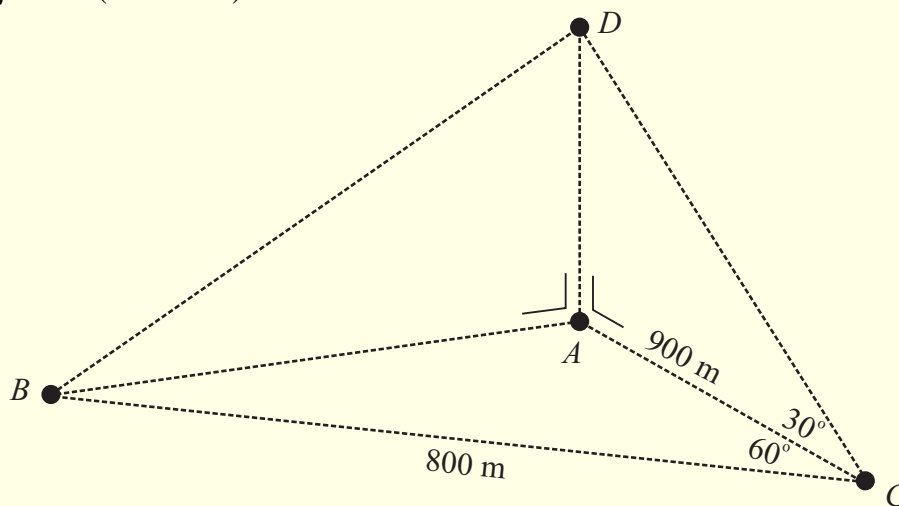
$$\alpha + \beta = 90^\circ \Rightarrow \beta = (90^\circ - \alpha)$$

$$\frac{\tan 2\alpha}{\tan 2\beta} = \frac{\tan 2\alpha}{\tan 2(90^\circ - \alpha)} = \frac{\tan 2\alpha}{\tan(180^\circ - 2\alpha)}$$

$$\tan(180^\circ - 2\alpha) = \frac{\tan 180^\circ - \tan 2\alpha}{1 + \tan 180^\circ \tan 2\alpha} = \frac{0 - \tan 2\alpha}{1 + (0) \tan 2\alpha} = -\tan 2\alpha$$

$$\therefore \frac{\tan 2\alpha}{\tan 2\beta} = \frac{\tan 2\alpha}{\tan(180^\circ - 2\alpha)} = \frac{\tan 2\alpha}{-\tan 2\alpha} = -1$$

5 (c) (i)

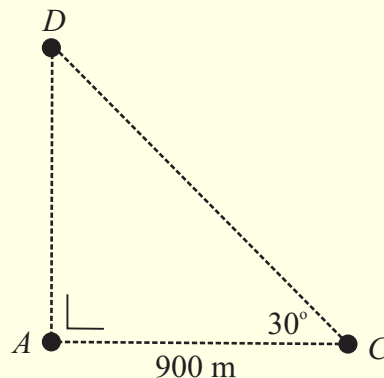


Lift out the right-angled triangle DAC:

$$\tan 30^\circ = \frac{|AD|}{900}$$

$$\therefore |AD| = 900 \tan 30^\circ$$

$$= 900 \left(\frac{1}{\sqrt{3}} \right) = 300\sqrt{3} \text{ m}$$



5 (c) (ii)

Apply the Cosine rule to triangle ABC :

$$\begin{aligned} |AB|^2 &= 900^2 + 800^2 - 2(900)(800) \cos 60^\circ \\ &= 810000 + 640000 - 2(900)(800)\left(\frac{1}{2}\right) \\ &= 810000 + 640000 - 720000 \\ &= 730000 \end{aligned}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\therefore |AB| = \sqrt{730000} \text{ m}$$

Lift out the right-angled triangle ABD and apply Pythagoras:

$$\begin{aligned} |BD|^2 &= |AB|^2 + |AD|^2 \\ &= (\sqrt{730000})^2 + (300\sqrt{3})^2 \\ &= 730000 + 90000 \times 3 \\ &= 730000 + 270000 \\ &= 1000000 \end{aligned}$$

$$\therefore |BD| = \sqrt{1000000} = 1000 \text{ m}$$

