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ALGEBRA (Q 1 & 2, PAPER 1)

2007

1 (a) Simplify $\frac{x^2 - xy}{x^2 - y^2}$.

(b) Let $f(x) = x^2 + (k+1)x - k - 2$, where k is a constant.

(i) Find the value of k for which $f(x) = 0$ has equal roots.

(ii) Find, in terms of k, the roots $f(x) = 0$.

(iii) Find the range of values of k for which both roots are positive.

(c) $x + p$ is a factor of both $ax^2 + b$ and $ax^2 + bx - ac$.

(i) Show that $p^2 = -\frac{b}{a}$ and that $p = \frac{-b - ac}{b}$.

(ii) Hence show that $p^2 + p^3 = c$.

SOLUTIONS

1 (a)

$$\frac{x^2 - xy}{x^2 - y^2} = \frac{x(x-y)}{(x+y)(x-y)} = \frac{x}{x+y}$$

$a^2 - b^2 = (a+b)(a-b)$ 1

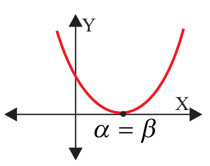
1 (b) (i)

$a = 1$
$b = (k+1)$
$c = -k - 2$

Equal roots $\Rightarrow b^2 = 4ac$

$\therefore (k+1)^2 = 4(-k-2) \Rightarrow k^2 + 2k + 1 = -4k - 8$

$k^2 + 6k + 9 = 0 \Rightarrow (k+3)(k+3) = 0 \Rightarrow k = -3$



$b^2 - 4ac = 0 \Rightarrow \alpha = \beta$
[2 equal real roots.]

1 (b) (ii)

$$x = \frac{-(k+1) \pm \sqrt{(k+1)^2 - 4(-k-2)}}{2} = \frac{-(k+1) \pm \sqrt{(k+3)^2}}{2}$$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ 4

$$= \frac{-(k+1) \pm (k+3)}{2} = \frac{-k - 1 + k + 3}{2}, \frac{-k - 1 - k - 3}{2}$$

$= 1, -k - 2$

1 (b) (iii)

Value of k for which both roots are positive:

$-k - 2 > 0 \Rightarrow -2 > k \Rightarrow k < -2$

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1 (c) (i)

You can prove these results by the division process or by lining up the coefficients.

Division Process

$$\begin{array}{r|l} & ax - ap \\ \hline x+p & ax^2 + b \\ & \mp ax^2 \mp apx \\ \hline & -apx + b \\ & \pm apx \pm ap^2 \\ \hline & ap^2 + b \end{array}$$

$$ap^2 + b = 0 \Rightarrow p^2 = -\tfrac{b}{a}$$

$$\begin{array}{r|l} & ax + (b - ap) \\ \hline x+p & ax^2 + bx - ac \\ & \mp ax^2 \mp apx \\ \hline & (b-ap)x - ac \\ & \mp(b-ap)x \mp p(b-ap) \\ \hline & -ac - p(b-ap) \end{array}$$

$$-ac - p(b-ap) = 0 \Rightarrow -ac = pb - ap^2$$

[Substitute $p^2 = -\tfrac{b}{a}$]

$$\Rightarrow -ac = pb + b \Rightarrow p = \frac{-ac-b}{b}$$

Lining up coefficients

$ax^2 + b = (x+p)(ax+k)$

Let $x = -p \Rightarrow ap^2 + b = 0 \Rightarrow p^2 = -\tfrac{b}{a}$ [This is an identity which is true for all values of x.]

$ax^2 + bx - c = (x+p)(ax+k)$

Let $x = -p \Rightarrow ap^2 - bp - c = 0 \Rightarrow -b - bp - ac = 0$ [Substitute $p^2 = -\tfrac{b}{a}$]

$$\therefore -b - ac = bp \Rightarrow p = \frac{-b-ac}{b}$$

1 (c) (ii)

$p = \dfrac{-b-ac}{b} = -1 - \dfrac{a}{b}c = -1 + \dfrac{c}{p^2}$ [Using $p^2 = -\tfrac{b}{a}$.]

$$\Rightarrow p = \frac{-p^2 + c}{p^2} \Rightarrow p^3 = -p^2 + c \Rightarrow p^2 + p^3 = c$$

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2. (a) Solve the simultaneous equations

$$x + y + z = 2$$
$$2x + y + z = 3$$
$$x - 2y + 2x = 15$$

(b) α and β are the roots of the equation $x^2 - 4x + 6 = 0$.

(i) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

(ii) Find the quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

(c) (i) Prove that $x + \dfrac{9}{x+2} \geq 4$, where $x + 2 > 0$.

(ii) Prove that $x + \dfrac{9}{x+a} \geq 6 - a$, where $x + a > 0$.

SOLUTION

2 (a)

STEPS
1. Eliminate one letter using two equations.
2. Eliminate the same letter using two others.
3. Solve the resulting equations as for two equations in two unknowns.
4. Work backwards to find all letters.

$$x + y + z = 2........(1)$$
$$2x + y + z = 3......(2)$$
$$x - 2y + 2z = 15..(3)$$

$$2x + y + z = 3......(2)$$
$$-x - y - z = -2...(1) \times (-1)$$
$$x = 1....(\mathbf{4})$$

$$x - 2y + 2z = 15...(\mathbf{3})$$
$$-2x - 2y - 2z = -4..(\mathbf{1}) \times (-2)$$
$$-x - 4y = 11....(\mathbf{5})$$

$$x = 1....(\mathbf{4})$$
$$-x - 4y = 11....(\mathbf{5})$$

→ SOLUTIONS: $x = 1, y = -3, z = 4$

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2 (b) (i)

$x^2 - 4x + 6 = 0$

$\mathbf{S}: \alpha + \beta = 4$

$\mathbf{P}: \alpha\beta = 6$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{4}{6} = \frac{2}{3}$$

[B] **Properties of roots** α, β

Sum S: $\alpha + \beta = -\frac{b}{a} = \frac{-2^{\text{nd.}}}{1^{\text{st.}}}$ 5

Product P: $\alpha\beta = \frac{c}{a} = \frac{3^{\text{rd.}}}{1^{\text{st.}}}$ 6

Forming a quadratic equation given its roots:

$x^2 - \mathbf{S}x + \mathbf{P} = 0$ 7

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

2 (b) (ii)

Quadratic Given

Roots: α, β

Sum $\mathbf{S}: \alpha + \beta = 4$

Product $\mathbf{P}: \alpha\beta = 6$

New Quadratic

Roots: $\frac{1}{\alpha}, \frac{1}{\beta}$

Sum **S**: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{2}{3}$

Product **P**: $\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{6}$

Quadratic: $x^2 - \frac{2}{3}x + \frac{1}{6} = 0 \Rightarrow 6x^2 - 4x + 1 = 0$

2 (c) (i)

$x + \frac{9}{x+2} \geq 4$ [You can multiply across by (x + 2) becuse you are told it is positive.]

$x(x+2) + 9 - 4(x+2) \geq 0 \Rightarrow x^2 + 2x + 9 - 4x - 8 \geq 0$

$\Rightarrow x^2 - 2x + 1 \geq 0 \Rightarrow (x-1)(x-1) \geq 0 \Rightarrow (x-1)^2 \geq 0$ [This is true].

2 (c) (ii)

$x + \frac{9}{x+a} \geq 6 - a$ [You can multiply across by (x + a) becuse you are told it is positive.]

$\Rightarrow x(x+a) + 9 - (x+a)(6-a) \geq 0$

$\Rightarrow x^2 + ax + 9 - 6x + ax - 6a + a^2 \geq 0$

$\Rightarrow x^2 + (2a-6)x + 9 - 6a + a^2 \geq 0$

$\Rightarrow x^2 + (2a-6)x + (a-3)^2 \geq 0$

$\Rightarrow [x + (a-3)][x + (a-3)] \geq 0$

$\Rightarrow [x + (a-3)]^2 \geq 0$ [This is true].