SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

LESSON NO. 6: BINOMIAL THEOREM

2004

4 (a) Show that $3\binom{n}{3} = n\binom{n-1}{2}$ for all natural numbers $n \geq 3$.

SOLUTION

4 (a)

To prove: $3\binom{n}{3} = n\binom{n-1}{2}$

LHS

$$3\binom{n}{3} = 3\times\frac{n(n-1)(n-2)}{3\times2\times1} = \frac{n(n-1)(n-2)}{2}$$

RHS

$$n\binom{n-1}{2} = n\times\frac{(n-1)(n-2)}{2}$$

2005

4 (b) (i) The first three terms in the binomial expansion of $(1+kx)^n$ are $1-21x+189x^2$. Find the value of *n* and the value of *k*.

SOLUTION

$$(1+kx)^n = \binom{n}{0}(1)^n(kx)^0 + \binom{n}{1}(1)^{n-1}(kx)^1 + \binom{n}{2}(1)^{n-2}(kx)^2 + ... = 1-21x+189x^2+...$$

$$\Rightarrow 1+nkx+\frac{n(n-1)}{2}k^2x^2+.... = 1-21x+189x^2+...$$

Lining up coefficients: $nk=-21 \Rightarrow k=-\frac{21}{n}$ and $\frac{n(n-1)}{2}k^2=189$

$$\Rightarrow 441n-441=378n \Rightarrow 63n=441 \Rightarrow n=7 \Rightarrow k=-3$$

2002

5 (b) The first three terms in the binomial expansion of $(1+ax)^n$ are $1+2x+\frac{7}{4}x^4$.

(i) Find the value of *a* and the value of *n*.

(ii) Hence, find the middle term in the expansion.

SOLUTION

5 (b) (i)

You need to remember the formula for expanding binomials, especially the first three terms.

$$(x+y)^n = \binom{n}{0}(x)^n(y)^0 + \binom{n}{1}(x)^{n-1}(y)^1 + \binom{n}{2}(x)^{n-2}(y)^2 \ldots\ldots\ldots\ldots\ 9$$

$$(1+ax)^n = 1+2x+\tfrac{7}{4}x^2+\ldots \Rightarrow \binom{n}{0}(1)^n(ax)^0 + \binom{n}{1}(1)^{n-1}(ax)^1 + \binom{n}{2}(1)^{n-2}(ax)^2 = 1+2x+\tfrac{7}{4}x^2$$

$$\Rightarrow 1+nax+\frac{n(n-1)}{2}a^2x^2 = 1+2x+\tfrac{7}{4}x^2$$

Lining up coefficients: $na = 2 \Rightarrow a = \frac{2}{n}$

$$\frac{n(n-1)}{2}a^2 = \frac{7}{4} \Rightarrow \frac{n(n-1)}{2}\left(\frac{2}{n}\right)^2 = \frac{7}{4} \Rightarrow \frac{n(n-1)}{2}\times\frac{4}{n^2} = \frac{7}{4}$$

$$\Rightarrow (n-1)\times\frac{2}{n} = \frac{7}{4} \Rightarrow 8(n-1) = 7n \Rightarrow 8n-8 = 7n$$

$$\Rightarrow n = 8 \Rightarrow a = \tfrac{2}{n} = \tfrac{2}{8} = \tfrac{1}{4}$$

5 (b) (ii)

Middle term of $(1+\frac{1}{4}x)^8$ is

$$\binom{8}{4}(1)^4(\tfrac{1}{4}x)^4 = \frac{70x^4}{256} = \frac{35x^4}{128}$$

The middle term is given by $\binom{n}{\frac{n}{2}}x^{\frac{n}{2}}y^{\frac{n}{2}}$.

2006

5 (a) Find the value of the middle term of the binomial expansion of $\left(\frac{x}{y}-\frac{y}{x}\right)^8$.

SOLUTION

5 (a)

$n = 8$

Middle term: $\binom{8}{4}\left(\frac{x}{y}\right)^4\left(-\frac{y}{x}\right)^4 = 70$

NOTE: The middle term is given by $\binom{n}{\frac{n}{2}}x^{\frac{n}{2}}y^{\frac{n}{2}}$.

2004

5 (a) Find the fifth term in the expansion of $\left(x^2-\frac{1}{x}\right)^6$ and show that it is independent of x.

SOLUTION

5 (a)

$u_5=?,\ r=4,\ n=6$

$$u_{r+1}={}^nC_r(x)^{n-r}(y)^r=\binom{n}{r}(x)^{n-r}(y)^r \quad \text{.......} \ \mathbf{10}$$

$$u_5=\binom{6}{4}(x^2)^2\left(-\frac{1}{x}\right)^4=15$$

2003

5 (c) Consider the binomial expansion of $\left(ax+\frac{1}{bx}\right)^8$, where a and b are non-zero real numbers.

(i) Write down the general term.

(ii) Given that the coefficient of x^2 is the equal to the coefficient of x^4, show that $ab = 2$.

SOLUTION

5 (c) (i)

General term of $\left(ax+\frac{1}{bx}\right)^8$

$$u_{r+1}={}^nC_r(x)^{n-r}(y)^r=\binom{n}{r}(x)^{n-r}(y)^r \quad \text{.......} \ \mathbf{10}$$

$n = 8,\ r = ?$

$$u_{r+1}=\binom{8}{r}(ax)^{8-r}\left(\frac{1}{bx}\right)^r$$

5 (c) (ii)

Tidy up the answer above $\Rightarrow u_{r+1}=\binom{8}{r}\frac{a^{8-r}x^{8-r}}{b^r x^r}=\binom{8}{r}\left(\frac{a^{8-r}}{b^r}\right)x^{8-2r}$

x^2 term: $8-2r=2\Rightarrow r=3$

x^4 term: $8-2r=4\Rightarrow r=2$

Coefficient of x^2 = Coefficient of $x^4\Rightarrow \binom{8}{3}\left(\frac{a^5}{b^3}\right)=\binom{8}{2}\left(\frac{a^6}{b^2}\right)$

$$\Rightarrow 56\left(\frac{a^5}{b^3}\right)=28\left(\frac{a^6}{b^2}\right)\Rightarrow 2\left(\frac{1}{b}\right)=a\Rightarrow ab=2$$

2001

5 (b) (ii) In the binomial expansion of $(1+kx)^6$, the coefficient of x^4 is 240. Find the two possible values of k.

SOLUTION

Write the general term of $(1+kx)^6$.

$$u_{r+1} = {}^nC_r(x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r \quad \ldots\ldots\ \mathbf{10}$$

$$u_{r+1} = \binom{6}{r}(1)^{6-r}(kx)^r = \binom{6}{r}k^r x^r$$

As can be seen, $r = 4$ in the term with x^4.

$$\Rightarrow \binom{6}{4}k^4 = 240 \Rightarrow 15k^4 = 240 \Rightarrow k^4 = 16 \Rightarrow k = \pm 2$$