Discrete Maths (Q 6 & 7, Paper 2)

1999

6 (a) In how many ways can a group of five people be selected from four women and four men?
In how many of these groups are there exactly three women?

(b) Solve the difference equation

$$u_{n+2} - 2u_{n+1} - 6u_n = 0, \text{ where } n \geq 0,$$

given that $u_0 = 0$ and $u_1 = 14$.

(c) In a class of 24 students, there are 14 boys and 10 girls.
In a particular week (Monday to Sunday inclusive), three students celebrate their birthdays. Assume that the birthdays are equally likely to fall on any day of the week and that the birthdays are independent of each other.
What is the probability that these three students

(i) are three boys or three girls

(ii) have birthdays falling on different days of the week or on the same day of the week other than Monday?

Solution

6 (a)

In how many ways can you pick 5 people from 8 people?

4W 4M
↓
5 People

The number of selections of n different objects taking r at a time $= {}^nC_r$ 11

$${}^8C_5 = {}^8C_3 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

In how many ways can you pick 5 people made up of 3 women and 2 men from 8 people?

4W 4M
↓ ↓
3W 2M

$${}^4C_3 \times {}^4C_2 = 4 \times 6 = 24$$

6 (b)

$u_n = l(\alpha)^n + m(\beta)^n$ 1

Steps

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $u_{n+2} - 2u_{n+1} - 6u_n = 0$

2. $x^2 - 2x - 6 = 0$

3. $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)} = \dfrac{2 \pm \sqrt{28}}{2}$

$\Rightarrow x = \dfrac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$

$\therefore \alpha = 1 + \sqrt{7},\ \beta = 1 - \sqrt{7}$

4. $u_n = l(1+\sqrt{7})^n + m(1-\sqrt{7})^n$

5. $u_0 = 0 \Rightarrow l + m = 0 \Rightarrow m = -l ...(\mathbf{1})$

$u_1 = 14 \Rightarrow l(1+\sqrt{7}) + m(1-\sqrt{7}) = 14 ...(\mathbf{2})$

Substitute Eqn. (**1**) into Eqn. (**2**):

$\therefore l(1+\sqrt{7}) - l(1-\sqrt{7}) = 14 \Rightarrow l + \sqrt{7}l - l + \sqrt{7}l = 14$

$\Rightarrow 2\sqrt{7}l = 14$

$\therefore l = \sqrt{7}$

$\therefore m = -\sqrt{7}$

Answer: $u_n = \sqrt{7}(1+\sqrt{7})^n - \sqrt{7}(1-\sqrt{7})^n$

6 (c) (i)

All you are being asked here is what is the probability of randomly picking 3 boys from 14 boys and 10 girls OR of randomly picking 3 girls from 14 boys and 10 girls. Their birthdays have nothing to do with this question.

OR means add the two probabilities together.
First, work out the probability of picking 3 boys.

The probability of picking the first boy is $\frac{14}{24}$ as there are 14 boys to pick from 24 students.

The probability of picking the second boy is $\frac{13}{23}$ as there is one less boy and one less student from which to pick.

The probability of picking the third boy is $\frac{12}{22}$.
Use the same procedure for the girls and add the two probabilities together.

$p(\text{3 boys } \textit{OR} \text{ 3 Girls}) = \frac{14}{24} \times \frac{13}{23} \times \frac{12}{22} + \frac{10}{24} \times \frac{9}{23} \times \frac{8}{22} = \frac{11}{46}$

6 (c) (ii)

The probability of a person having their birthday on some day of the week is $\frac{7}{7}$, i.e. it is certain that your birthday falls on some day of the week.
The probability of a second person having a birthday on a day of the week different to the first person is $\frac{6}{7}$. There are 6 days to pick from the 7 days of the week.

The probability of a third person having a birthday different to the other 2 people is $\frac{5}{7}$.

$p(\text{3 birthdays on different days}) = \frac{7}{7} \times \frac{6}{7} \times \frac{5}{7} = \frac{210}{343}$

The probability of a person having their birthday on some day of the week other than Monday is $\frac{6}{7}$.
The probability of a second person having a birthday on the same day of the week as the first person is $\frac{1}{7}$.
The probability of a third person having a birthday on the same day as the other two students is $\frac{1}{7}$.

p(3 birthdays on the same day other than Monday) $= \frac{6}{7} \times \frac{1}{7} \times \frac{1}{7} = \frac{6}{343}$

Therefore, the p(3 birthdays on different days OR 3 birthdays on the same day other than Monday) $= \frac{210}{343} + \frac{6}{343} = \frac{216}{343}$

7 (a) Six discs of equal size are stacked one on top of the other. There are two identical red discs and one each of blue, yellow, green and white.

In how many different ways can the six discs be stacked so that the two red discs are either at the top or at the bottom?

(b) Two balls are at the same time taken at random from a box containing three black, three red and three yellow balls.
Find the probability that
(i) both balls are yellow
(ii) neither of the two balls is yellow
(iii) at least one of the two balls is yellow.

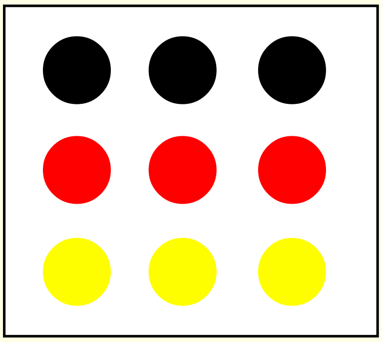
(c) The numbers a, $3a$, b, $2b$ have mean $2b$ and standard deviation σ.
(i) Express b in terms of a.
(ii) Express σ in terms of a.
(iii) Find the range of values of a for which $\sigma^2 < 18 \cdot 5$.

SOLUTION

7 (a)

The 2 identical red discs can be on top or on the bottom (2 ways).
NOTE: Changing the order of the 2 red discs does not give you another arrangement as they are identical.
The other 4 discs can be arranged in 4! ways.
Therefore, total number of arrangements $= 2 \times 4! = 48$

7 (b) (i)



$$p(E) = \frac{\text{Number of desired outcomes}}{\text{Total possible number of outcomes}} = \frac{\#(E)}{\#(S)}$$ 12

The probability of picking a yellow on the first pick is $\frac{3}{9}$ because there are 3 yellow balls to pick from 9 balls.

The probability of picking a second yellow is $\frac{2}{8}$ as there are now only 2 yellow balls to pick from 8 balls.

$p(\text{2 Yellows}) = \frac{3}{9} \times \frac{2}{8} = \frac{1}{12}$

7 (b) (ii)

The probability of not picking a yellow on the first pick is $\frac{6}{9}$ because there are 6 non-yellow balls to pick from 9 balls.

The probability of picking a second non-yellow is $\frac{5}{8}$ as there are now 5 non-yellow balls to pick from 8 balls.

$p(\text{2 Non-Yellows}) = \frac{6}{9} \times \frac{5}{8} = \frac{5}{12}$

7 (b) (iii)

At least one = 1 − p(None)
At least two = 1 − p(None or one) etc... 15

$p(\text{At least 1 yellow}) = 1 - \frac{5}{12} = \frac{7}{12}$

7 (c) (i)

$$\bar{x} = \frac{x_1 + x_2 + \ldots\ldots\ldots\ldots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N}$$ 3

$$\bar{x} = 2b = \frac{a + 3a + b + 2b}{4}$$

$\Rightarrow 8b = 4a + 3b$

$\Rightarrow 5b = 4a$

$\therefore b = \frac{4}{5}a$

7 (c) (ii)

x	d	d^2
a	$a-2b$	$(a-2b)^2$
$3a$	$3a-2b$	$(3a-2b)^2$
b	$-b$	b^2
$2b$	0	0

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}} \quad \text{.......} \; \textbf{6}$$

where deviation $d = (x - \overline{x}) = (\text{Number} - \text{Mean})$

$$\sigma = \sqrt{\frac{(a-2b)^2 + (3a-2b)^2 + b^2 + 0}{4}}$$

$$\Rightarrow \sigma = \sqrt{\frac{(a - \frac{8}{5}a)^2 + (3a - \frac{8}{5}a)^2 + (\frac{4}{5}a)^2}{4}}$$

$$\Rightarrow \sigma = \sqrt{\frac{(-\frac{3}{5}a)^2 + (\frac{7}{5}a)^2 + (\frac{4}{5}a)^2}{4}}$$

$$\Rightarrow \sigma = \sqrt{\frac{\frac{9}{25}a^2 + \frac{49}{25}a^2 + \frac{16}{25}a^2}{4}} = \sqrt{\frac{\frac{74}{25}a^2}{4}} = \sqrt{\frac{74a^2}{100}}$$

$$\therefore \sigma = \frac{\sqrt{74}a}{10}$$

7 (c) (iii)

$$\sigma^2 < 18.5$$

$$\Rightarrow \frac{74a^2}{100} < 18.5$$

$$\Rightarrow 74a^2 < 1850$$

$$\Rightarrow a^2 < 25$$

$$\therefore -5 < a < 5$$