

**DISCRETE MATHS (Q 6 & 7, PAPER 2)**

**LESSON NO. 1: DIFFERENCE EQUATIONS**

**2006**

6 (b) (i) Solve the difference equation  $6u_{n+2} - 7u_{n+1} + u_n = 0$ , where  $n \geq 0$ , given that  $u_0 = 8$  and  $u_1 = 3$ .

(ii) Verify that the solution to part (i) also satisfies the difference equation

$$6u_{n+1} - u_n - 10 = 0.$$

**SOLUTION**

**6 (b) (i)**

$$u_n = l(\alpha)^n + m(\beta)^n \dots\dots \textcircled{1}$$

**STEPS**

1. Write the Second Order Difference Equation in decreasing order of subscripts:  $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation:  $px^2 + qx + r = 0$
3. Solve this equation to find  $\alpha, \beta$ .
4. Write solution as:  $u_n = l(\alpha)^n + m(\beta)^n$
5. Find  $l, m$  using extra conditions (boundary conditions).

1.  $6u_{n+2} - 7u_{n+1} + u_n = 0$

2.  $6x^2 - 7x + 1 = 0$

3.  $(x-1)(6x-1) = 0 \Rightarrow \alpha = 1, \beta = \frac{1}{6}$

4.  $u_n = l(1)^n + m(\frac{1}{6})^n = l + m(\frac{1}{6})^n$

5.  $u_0 = 8 = l + m$  and  $u_1 = 3 = l + m(\frac{1}{6}) \Rightarrow 18 = 6l + m$

Solving these equations simultaneously  $\Rightarrow l = 2, m = 6$

**Ans:**  $u_n = 2 + 6(\frac{1}{6})^n = 2 + (\frac{1}{6})^{n-1}$

**6 (b) (ii)**

Substitute the solution into  $6u_{n+1} - u_n - 10 = 0$ .

$$u_n = 2 + (\frac{1}{6})^{n-1}$$

$$u_{n+1} = 2 + (\frac{1}{6})^n$$

$$\Rightarrow 6u_{n+1} - u_n - 10 = 6(2 + (\frac{1}{6})^n) - (2 + (\frac{1}{6})^{n-1}) - 10$$

$$= 12 + (\frac{1}{6})^{n-1} - 2 - (\frac{1}{6})^{n-1} - 10 = 0$$

2005

6 (b) (i) Solve the difference equation  $u_{n+2} - 4u_{n+1} - 8u_n = 0$ , where  $n \geq 0$ , given that  $u_0 = 0$  and  $u_1 = 2$ .

(ii) Verify that your solution gives the correct value for  $u_2$ .

**SOLUTION**

6 (b) (i)

$$u_n = l(\alpha)^n + m(\beta)^n \dots\dots \textcircled{1}$$

**STEPS**

1. Write the Second Order Difference Equation in decreasing order of subscripts:  $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation:  $px^2 + qx + r = 0$
3. Solve this equation to find  $\alpha, \beta$ .
4. Write solution as:  $u_n = l(\alpha)^n + m(\beta)^n$
5. Find  $l, m$  using extra conditions (boundary conditions).

1.  $u_{n+2} - 4u_{n+1} - 8u_n = 0$

2.  $x^2 - 4x - 8 = 0$

3.  $x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-8)}}{2} = \frac{4 \pm \sqrt{48}}{2} = \frac{4 \pm 4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}$

$$\Rightarrow \alpha = 2 + 2\sqrt{3}, \beta = 2 - 2\sqrt{3}$$

4.  $u_n = l(2 + 2\sqrt{3})^n + m(2 - 2\sqrt{3})^n$

5.  $u_0 = l + m = 0 \dots \textcircled{1}$  and  $u_1 = l(2 + 2\sqrt{3})^1 + m(2 - 2\sqrt{3})^1 = 2 \dots \textcircled{2}$

From equation (1)  $\Rightarrow m = -l$

Substitute into equation (2):  $\Rightarrow l(2 + 2\sqrt{3}) - l(2 - 2\sqrt{3}) = 2 \Rightarrow 2l + 2l\sqrt{3} - 2l + 2l\sqrt{3} = 2$

$$\Rightarrow 4l\sqrt{3} = 2 \Rightarrow l = \frac{1}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{6}$$

$$\therefore l = \frac{\sqrt{3}}{6}, m = -\frac{\sqrt{3}}{6}$$

**Ans:**  $u_n = \frac{\sqrt{3}}{6}(2 + 2\sqrt{3})^n - \frac{\sqrt{3}}{6}(2 - 2\sqrt{3})^n$

6 (b) (ii)

Put  $n = 0$  into the difference equation:  $u_2 - 4u_1 - 8u_0 = 0 \Rightarrow u_2 = 4(2) + 8(0) = 8$

Put  $n = 2$  into the solution:

$$u_2 = \frac{\sqrt{3}}{6}(2 + 2\sqrt{3})^2 - \frac{\sqrt{3}}{6}(2 - 2\sqrt{3})^2 \Rightarrow u_2 = \frac{\sqrt{3}}{6}(4 + 8\sqrt{3} + 12 - 4 + 8\sqrt{3} - 12)$$

$$\Rightarrow u_2 = \frac{\sqrt{3}}{6}(4 + 8\sqrt{3} + 12 - 4 + 8\sqrt{3} - 12) = \frac{\sqrt{3}}{6}(16\sqrt{3}) = 8$$

You have verified that you get the correct solution for  $u_2$ .

**2004**

6 (b) (i) Solve the difference equation  $3u_{n+2} - 2u_{n+1} - u_n = 0$ , where  $n \geq 0$ , given that

$$u_0 = 3 \text{ and } u_1 = 7.$$

(ii) Evaluate  $\lim_{n \rightarrow \infty} u_n$ .

**SOLUTION**

**6 (b) (i)**

$$u_n = l(\alpha)^n + m(\beta)^n \dots\dots \mathbf{1}$$

**STEPS**

1. Write the Second Order Difference Equation in decreasing order of subscripts:  $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation:  $px^2 + qx + r = 0$
3. Solve this equation to find  $\alpha, \beta$ .
4. Write solution as:  $u_n = l(\alpha)^n + m(\beta)^n$
5. Find  $l, m$  using extra conditions (boundary conditions).

1.  $3u_{n+2} - 2u_{n+1} - u_n = 0$

2.  $3x^2 - 2x - 1 = 0$

3.  $\therefore (3x+1)(x-1) = 0 \Rightarrow \alpha = 1, \beta = -\frac{1}{3}$

4.  $u_n = l(1)^n + m(-\frac{1}{3})^n = l + m(-\frac{1}{3})^n$

5.  $u_0 = 3 = l + m$  and  $u_1 = 7 = l + m(-\frac{1}{3}) \Rightarrow 21 = 3l - m$

Solving these equations simultaneously  $\Rightarrow l = 6, m = -3$

**Ans:**  $u_n = 6 - 3(-\frac{1}{3})^n$

**6 (b) (ii)**

You may be asked to find a limit to infinity.

Remember  $\lim_{n \rightarrow \infty} r^n = 0, -1 < r < 1$  but  $\lim_{n \rightarrow \infty} r^n = \infty, r < -1, r > 1$

$$\lim_{n \rightarrow \infty} u_n = 6$$

**2003**

6 (b) (i) Solve the difference equation  $u_{n+2} - 4u_{n+1} + 3u_n = 0$ , where  $n \geq 0$ , given that  $u_0 = -2$  and  $u_1 = 4$ .

(ii) Verify that the solution you have obtained in (i) satisfies the difference equation.

**SOLUTION**

**6 (b) (i)**

$$u_n = l(\alpha)^n + m(\beta)^n \dots\dots \textcircled{1}$$

**STEPS**

1. Write the Second Order Difference Equation in decreasing order of subscripts:  $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation:  $px^2 + qx + r = 0$
3. Solve this equation to find  $\alpha, \beta$ .
4. Write solution as:  $u_n = l(\alpha)^n + m(\beta)^n$
5. Find  $l, m$  using extra conditions (boundary conditions).

1.  $u_{n+2} - 4u_{n+1} + 3u_n = 0$

2.  $x^2 - 4x + 3 = 0$

3.  $(x-1)(x-3) = 0 \Rightarrow \alpha = 1, \beta = 3$

4.  $u_n = l(1)^n + m(3)^n = l + m(3)^n$

5.  $u_0 = -2 = l + m$  and  $u_1 = 4 = l + 3m$

Solving these equations simultaneously  $\Rightarrow l = -5, m = 3$

**Ans:**  $u_n = 3(3)^n - 5$

**6 (b) (ii)**

$$u_n = 3(3)^n - 5$$

$$u_{n+1} = 3(3)^{n+1} - 5$$

$$u_{n+2} = 3(3)^{n+2} - 5$$

$$\Rightarrow 3(3)^{n+2} - 5 - 4[3(3)^{n+1} - 5] + 3[3(3)^n - 5] = 3(3)^{n+2} - 5 - 12(3)^{n+1} + 20 + 9(3)^n - 15$$

$$= (3)^n [3(3)^2 - 12(3) + 9] = (3)^n [27 - 36 + 9] = (3)^n [0] = 0$$

**2002**

6 (b) (i) Solve the difference equation  $6u_{n+2} - 5u_{n+1} + u_n = 0$ , where  $n \geq 0$ , given that  $u_0 = 5$  and  $u_1 = 2$ .

(ii) Find an expression in  $n$  for the sum of the terms  $u_0 + u_1 + u_2 + \dots + u_n$ .  
(Hint: It is the sum of two geometric series.)

(iii) Evaluate the sum to infinity of this series (that is:  $\sum_{n=0}^{\infty} u_n$ ).

**SOLUTION**

**6 (b) (i)**

$$u_n = l(\alpha)^n + m(\beta)^n \dots\dots \mathbf{1}$$

**STEPS**

1. Write the Second Order Difference Equation in decreasing order of subscripts:  $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation:  $px^2 + qx + r = 0$
3. Solve this equation to find  $\alpha, \beta$ .
4. Write solution as:  $u_n = l(\alpha)^n + m(\beta)^n$
5. Find  $l, m$  using extra conditions (boundary conditions).

1.  $6u_{n+2} - 5u_{n+1} + u_n = 0$

2.  $6x^2 - 5x + 1 = 0$

3.  $(2x - 1)(3x - 1) = 0 \Rightarrow \alpha = \frac{1}{2}, \beta = \frac{1}{3}$

4.  $u_n = l(\frac{1}{2})^n + m(\frac{1}{3})^n$

5.  $u_0 = 5 = l + m$  and  $u_1 = 2 = \frac{1}{2}l + \frac{1}{3}m \Rightarrow 12 = 3l + 2m$

Solving these equations simultaneously  $\Rightarrow l = 2, m = 3$

**Ans:**  $u_n = 2(\frac{1}{2})^n + 3(\frac{1}{3})^n$

**6 (b) (ii)**

Summing formula:  $S_n = \frac{a(1-r^n)}{(1-r)}$  ..... **5**

$$u_n = 2(\frac{1}{2})^n + 3(\frac{1}{3})^n = 2[1 + (\frac{1}{2}) + (\frac{1}{2})^2 + \dots + (\frac{1}{2})^{n+1}] + 3[1 + (\frac{1}{3}) + (\frac{1}{3})^2 + \dots + (\frac{1}{3})^{n+1}]$$

$\uparrow$   $a = 1, r = \frac{1}{2}, S_{n+1}$                        $\uparrow$   $a = 1, r = \frac{1}{3}, S_{n+1}$

$$\Rightarrow u_n = \frac{2(1 - (\frac{1}{2})^{n+1})}{1 - \frac{1}{2}} + \frac{3(1 - (\frac{1}{3})^{n+1})}{1 - \frac{1}{3}} = 4(1 - (\frac{1}{2})^{n+1}) + \frac{9}{2}(1 - (\frac{1}{3})^{n+1})$$

**6 (b) (iii)**

$$\sum_{n=0}^{\infty} u_n = 4(1-0) + \frac{9}{2}(1-0) = \frac{17}{2}$$

**FAMOUS LIMITS**

$\lim_{n \rightarrow \infty} r^n = 0$  for  $-1 < r < 1$ . **Example:**  $\lim_{n \rightarrow \infty} (\frac{3}{5})^n = 0$

$\lim_{n \rightarrow \infty} r^n = \infty$  for  $r > 1, r < -1$ . **Example:**  $\lim_{n \rightarrow \infty} (\frac{3}{2})^n = \infty$

2001

6 (b) Solve the difference equation  $u_{n+2} - 8u_{n+1} + 11u_n = 0$ , where  $n \geq 0$ , given that  $u_0 = 0$  and  $u_1 = 2\sqrt{15}$ .

**SOLUTION**

6 (b)

$$u_n = l(\alpha)^n + m(\beta)^n \dots\dots \textcircled{1}$$

**STEPS**

1. Write the Second Order Difference Equation in decreasing order of subscripts:  $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation:  $px^2 + qx + r = 0$
3. Solve this equation to find  $\alpha, \beta$ .
4. Write solution as:  $u_n = l(\alpha)^n + m(\beta)^n$
5. Find  $l, m$  using extra conditions (boundary conditions).

1.  $u_{n+2} - 8u_{n+1} + 11u_n = 0$

2.  $x^2 - 8x + 11 = 0$

3.  $x = \frac{8 \pm \sqrt{64 - 4(1)(11)}}{2} = \frac{8 \pm \sqrt{20}}{2} = \frac{8 \pm 2\sqrt{5}}{2} = 4 \pm \sqrt{5}$

$\Rightarrow \alpha = 4 + \sqrt{5}, \beta = 4 - \sqrt{5}$

4.  $u_n = l(4 + \sqrt{5})^n + m(4 - \sqrt{5})^n$

5.  $u_0 = l + m = 0 \Rightarrow m = -l$  and  $u_1 = l(4 + \sqrt{5}) + m(4 - \sqrt{5}) = 2\sqrt{15}$

Solving these equations simultaneously:

$$l(4 + \sqrt{5}) - l(4 - \sqrt{5}) = 2\sqrt{15} \Rightarrow 2l\sqrt{5} = 2\sqrt{15}$$

$$\Rightarrow l = \sqrt{3}, m = -\sqrt{3}$$

**Ans:**  $u_n = \sqrt{3}(4 + \sqrt{5})^n - \sqrt{3}(4 - \sqrt{5})^n$