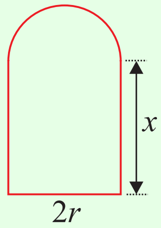
CALCULUS OPTION (Q 8, PAPER 2)

2011

8. **(a)** Use integration by parts to find $\int x \sin x \, dx$.

(b) A window is in the shape of a rectangle with a semicircle on top.
The radius of the semicircle is r metres and the height of the rectangular part is x metres. The perimeter of the window is 20 metres.



(i) Use the perimeter to express x in terms of r and π.

(ii) Find, in terms of π, the value of r for which the area of the window is a maximum.

(c) The Maclaurin series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(i) Write down the general term of the series.

(ii) Use the Ratio Test to show that the series converges for $|x| < 1$.

(iii) Using the fact that $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$, and taking the first three terms in the Maclaurin series for $\tan^{-1} x$, find an approximation for π.
Give your answer correct to five decimal places.

SOLUTION

8 (a)

PARTS FORMULA $\int u \, dv = uv - \int v \, du$

STEPS
1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation;
   Let dv equal what is left and find v by integration.
   NOTE: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v \, du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

LIST of Functions
1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I = \int x \sin x \, dx$

2.

$u = x$	$dv = \sin x \, dx$
$du = 1 \, dx$	$v = -\cos x$

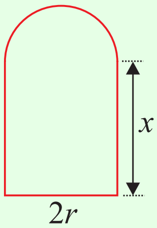
3. $I = uv - \int v du = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x + c$

8 (b)

Steps
1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: *V* for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. *A* (Area)

2. Diagram



3. $A = 2rx + \frac{1}{2}\pi r^2$

4. Extra information: Perimeter $P = 20$ m

$$20 = x + 2r + x + \pi r$$
$$20 = 2x + 2r + \pi r$$
$$20 - 2r - \pi r = 2x$$
$$\therefore x = \frac{20 - 2r - \pi r}{2}$$

5. $A = 2rx + \frac{1}{2}\pi r^2$

$$= 2r\left(\frac{20 - 2r - \pi r}{2}\right) + \frac{1}{2}\pi r^2$$
$$= r(20 - 2r - \pi r) + \frac{1}{2}\pi r^2$$
$$= 20r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$$
$$= 20r - 2r^2 - \frac{1}{2}\pi r^2$$

6. $\frac{dA}{dr} = 0 \Rightarrow 20 - 4r - \pi r = 0$

$$20 = \pi r + 4r$$
$$20 = r(\pi + 4)$$
$$\therefore \frac{20}{(\pi + 4)} = r$$

Answers

8 (b) (i) $\frac{20 - 2r - \pi r}{2}$

8 (b) (ii) $\frac{20}{\pi + 4}$

8 (c) (i)

Steps

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

$$T_n = a + (n-1)d$$

2. Sometimes the signs alternate: $+, -, +, -, +, -$....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

$$\tan^{-1} x = \frac{x^1}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

Powers, Fractions: 1, 3, 5, 7...... [$a = 1$, $d = 2$]

$$T_n = 1 + (n-1)2 = 1 + 2n - 2 = 2n - 1$$

$$\therefore u_n = \frac{x^{2n-1}}{2n-1}(-1)^{n-1}$$

8 (c) (ii)

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$.

Steps

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = \frac{x^{2n-1}}{2n-1}(-1)^{n-1}$

2. $u_{n+1} = \frac{x^{2(n+1)-1}}{2(n+1)-1}(-1)^{(n+1)-1} = \frac{x^{2n+1}}{2n+1}(-1)^n$

3.
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n+1}}{(2n+1)} \times \frac{(2n-1)}{(-1)^{n-1}x^{2n-1}}\right|$$

$$= \lim_{n\to\infty}\left|(-1)x^2\frac{(2n-1)}{(2n+1)}\right|$$

$$= \lim_{n\to\infty}\left|(-1)x^2\frac{n(2-\frac{1}{n})}{n(2+\frac{1}{n})}\right|$$

$$= \lim_{n\to\infty}\left|(-1)x^2\frac{(2)}{(2)}\right|$$

$$= \left|x^2\right|$$

Series is convergent for $\left|x^2\right| < 1 \Rightarrow -1 < x < 1$

8 (c) (iii)

$$4\tan^{-1}(\tfrac{1}{5}) = 4(\tfrac{1}{5} - \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5) = 0.789589$$

$$\tan^{-1}(\tfrac{1}{239}) = \tfrac{1}{239} - \tfrac{1}{3}(\tfrac{1}{239})^3 + \tfrac{1}{5}(\tfrac{1}{239})^5) = 0.004184$$

$$\therefore 4\tan^{-1}(\tfrac{1}{5}) + \tan^{-1}(\tfrac{1}{239}) = 0.789589 + 0.004184 = 0.793773$$

$$\tfrac{\pi}{4} \approx 0.793773 \Rightarrow \pi = 4(0.793773) = 3.17509$$