CALCULUS OPTION (Q 8, PAPER 2)

2007

8 (a) p and q are real numbers such that $p + q = 1$.
Find the value of p that maximizes the product pq.

(b) (i) Derive the Maclaurin series for $f(x) = (1+x)^m$ up to an including the term containing x^3.

(ii) Given that the general term of the series $f(x)$ is

$$\frac{m(m-1)(m-2).......(m-r+1)}{r!}x^r,$$

show that the series converges for $-1 < x < 1$.

(c) Evaluate $\int_0^1 \tan^{-1} x\,dx$.

SOLUTION

8 (a)

> **STEPS**
> **1**. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: *V* for volume.
> **2**. Draw a diagram (if necessary) and put in the variable(s).
> **3**. Write the quantity in terms of this/these variable(s).
> **4**. If there are 2 variables get rid of one in terms of the other using extra information.
> **5**. Hence, write the quantity as a function of a single variable.
> **6**. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
> **7**. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Product, P

2. No diagram needed.

3. $P = pq$

4. $p + q = 1 \Rightarrow q = 1 - p$ [Extra information]

5. $P = p(1-p) = p - p^2$

6. $\frac{dP}{dp} = 1 - 2p = 0 \Rightarrow p = \frac{1}{2}$

8 (b) (i)

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ **3**

$f(x)=(1+x)^m \Rightarrow f(0)=1$

$f'(x)=m(1+x)^{m-1} \Rightarrow f'(0)=m$

$f''(x)=m(m-1)(1+x)^{m-2} \Rightarrow f''(0)=m(m-1)$

$f'''(x)=m(m-1)(m-2)(1+x)^{m-3} \Rightarrow f'''(0)=m(m-1)(m-2)$

$$\Rightarrow (1+x)^m=1+\frac{mx}{1!}+\frac{m(m-1)x^2}{2!}+\frac{m(m-1)(m-2)x^3}{3!}$$

$$\Rightarrow (1+x)^m=\binom{m}{0}+\binom{m}{1}x+\binom{m}{2}x^2+\binom{m}{3}x^3$$

8 (b) (ii)

$\sum_{n=1}^{\infty}u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|>1$. **2**

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty}u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|>1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=1$ the test is **inconclusive**.

1. $u_r=\frac{m(m-1)(m-2).......(m-r+1)}{r!}x^r$

2. $u_{r+1}=\frac{m(m-1)(m-2).......(m-r)}{(r+1)!}x^{r+1}$

3. $$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right|=\lim_{r\to\infty}\left|\frac{m(m-1)(m-2).......(m-r)x^{r+1}}{(r+1)!}\times\frac{r!}{m(m-1)(m-2).......(m-r+1)x^r}\right|$$

$$=\lim_{r\to\infty}\left|\frac{x(m-r)}{(r+1)}\right|=\lim_{r\to\infty}\left|\frac{x\not{r}(\frac{m}{r}-1)}{\not{r}(1+\frac{1}{r})}\right|=|x|$$

Series converges for $|x|<1 \Rightarrow -1<x<1$

8 (c)

$\int u\,dv = uv - \int v\,du$ .... **1** This formula is on page 42 of the tables.

STEPS
1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation; Let dv equal what is left and find v by integration.
   NOTE: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v\,du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

1. $I = \int \tan^{-1} x\,dx$

2.

$u = \tan^{-1} x \qquad dv = 1dx$

$du = \dfrac{1}{1+x^2}dx \qquad v = x$

3. $\therefore I = x\tan^{-1} x - \int \dfrac{x}{1+x^2}dx$

Find $\int \dfrac{x}{1+x^2}dx$ by substitution.

Let $u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$\int \dfrac{x}{1+x^2}dx = \frac{1}{2}\int \dfrac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(1+x^2)+c$

$\therefore I = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)+c$

4. $\therefore I = [x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)]_0^1 = [(1\tan^{-1} 1 - \frac{1}{2}\ln(1+1)) - (0\tan^{-1} 0 - \frac{1}{2}\ln(1+0))]$

$= [(\frac{\pi}{4} - \frac{1}{2}\ln 2) - (0-0)] = \frac{\pi}{4} - \frac{1}{2}\ln 2$