

LINE (Q 3, PAPER 2)

LESSON NO. 6: LINEAR TRANSFORMATIONS

2003

3 (a) f is the transformation $(x, y) \rightarrow (x', y')$ where $x' = x + y$ and $y' = x - y$. L is the line $4x - 2y - 1 = 0$. Find the equation of $f(L)$, the image of L under f .

SOLUTION

3 (a)

$$\begin{array}{rcl} x' = x + y & \Leftarrow x' = x + y \Rightarrow & x = x + y \\ y' = x - y & \Leftarrow y' = x - y \Rightarrow & -y' = -x + y \\ \hline \frac{x' + y'}{2} = x & & \frac{x' - y'}{2} = y \end{array}$$

$$f(L): 4x - 2y - 1 = 0 \Rightarrow 4\left(\frac{x' + y'}{2}\right) - 2\left(\frac{x' - y'}{2}\right) - 1 = 0$$

$$\Rightarrow 2(x' + y') - (x' - y') - 1 = 0 \Rightarrow 2x' + 2y' - x' + y' - 1 = 0$$

$$\Rightarrow x' + 3y' - 1 = 0$$

2002

3 (b) f is the transformation $(x, y) \rightarrow (x', y')$ where $x' = 3x + y$ and $y' = x - 2y$. S is the square whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

- (i) Find the image of f of each of the four vertices of S .
- (ii) Express x and y in terms of x' and y' .
- (iii) By considering the lines $ax + by + c = 0$ and $ax + by + d = 0$, or otherwise, prove that f maps every pair of parallel lines. (You may assume that f maps every line to a line.)
- (iv) Show both S and $f(S)$ on a diagram.
- (v) Find the area of $f(S)$.

SOLUTION

3 (b) (i)

$$(0, 0) \rightarrow (0, 0): x' = 3x + y = 3(0) + (0) = 0, y' = x - 2y = (0) - 2(0) = 0$$

$$(1, 0) \rightarrow (3, 1): x' = 3x + y = 3(1) + (0) = 3, y' = x - 2y = (1) - 2(0) = 1$$

$$(1, 1) \rightarrow (4, -1): x' = 3x + y = 3(1) + (1) = 4, y' = x - 2y = (1) - 2(1) = -1$$

$$(0, 1) \rightarrow (1, -2): x' = 3x + y = 3(0) + (1) = 1, y' = x - 2y = (0) - 2(1) = -2$$

3 (b) (ii)

$$\begin{array}{lcl}
 2x' = 6x + 2y & \Leftrightarrow x' = 3x + y \Rightarrow & x' = 3x + y \\
 y' = x - 2y & \Leftrightarrow y' = x - 2y \Rightarrow & -3y' = -3x + 6y \\
 \hline
 \frac{2x' + y'}{7} = x & & \frac{x' - 3y'}{7} = y
 \end{array}$$

3 (b) (iii)

Call the two parallel lines $K: ax + by + c = 0$ and $L: ax + by + d = 0$.

$$K' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7c = 0$$

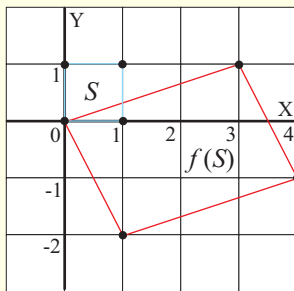
$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7c = 0$$

$$L' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7d = 0$$

$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7d = 0$$

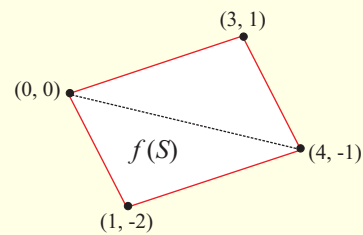
$$\therefore K \parallel L \Leftrightarrow K' \parallel L'$$

3 (b) (iv)



3 (b) (v)

Under linear transformations:
 1. Parallelograms \rightarrow Parallelograms
 $\square abcd \rightarrow \square a'b'c'd'$



The diagonal bisects the area of the parallelogram.

$$\therefore A = |(3)(-1) - (4)(1)| = |-3 - 4| = 7$$

$$A = \frac{1}{2} |x_1 y_2 - x_2 y_1| \dots \dots \textcircled{4}$$

2001

3 (b) f is the transformation $(x, y) \rightarrow (x', y')$

$$x' = -5x - 6y$$

$$y' = 4x + 3y.$$

L is the line $x - 9y = 2$.

(i) Find the equation of $f(L)$, the image of L under f .

M is a line containing the point $(1, k)$ where $k \in \mathbf{Z}$.

(ii) Given that $f(M)$ is $5x' - 2y' + 3k = 0$, find the value of k .

SOLUTION

3 (b) (i)

$$\begin{array}{rcl} x' = -5x - 6y & \Leftarrow x' = -5x - 6y \Rightarrow & 4x' = -20x - 24y \\ 2y' = 8x + 6y & \Leftarrow y' = 4x + 3y \Rightarrow & 5y' = 20x + 15y \\ \hline \frac{x' + 2y'}{3} = x & & -\frac{4x' + 5y'}{9} = y \end{array}$$

$$L: x - 9y - 2 = 0 \Rightarrow f(L): \left(\frac{x' + 2y'}{3}\right) - 9\left(-\frac{4x' + 5y'}{9}\right) - 2 = 0$$

$$\Rightarrow f(L): x' + 2y' + 12x' + 15y' - 6 = 0 \Rightarrow f(L): 13x' + 17y' - 6 = 0$$

3 (b) (ii)

$$f(M): 5x' - 2y' + 3k = 0 \Rightarrow M: 5(-5x - 6y) - 2(4x + 3y) + 3k = 0$$

$$\Rightarrow M: -25x - 30y - 8x - 6y + 3k = 0 \Rightarrow M: 33x + 36y - 3k = 0$$

$$\Rightarrow M: 11x + 12y - k = 0$$

$$(1, k) \in M \Rightarrow 11(1) + 12(k) - k = 0 \Rightarrow 11 + 12k - k = 0 \Rightarrow 11 + 11k = 0$$

$$\Rightarrow 11 = -11k \Rightarrow k = -1$$

2005

3 (c) f is the transformation $(x, y) \rightarrow (x', y')$, where $x' = 3x - y$ and $y' = x + 2y$.

- (i) Prove that f maps every pair of parallel lines to a pair of parallel lines. You may assume that f maps every line to a line.
- (ii) $oabc$ is a parallelogram, where $[ob]$ is a diagonal and o is the origin. Given that $f(c) = (-1, 9)$, find the slope of ab .

SOLUTION

3 (c) (i)

$$\begin{array}{lll} 2x' = 6x - 2y & \Leftrightarrow x' = 3x - y \Rightarrow & x' = 3x - y \\ y' = x + 2y & \Leftrightarrow y' = x + 2y \Rightarrow & -3y' = -3x - 6y \\ \hline \frac{2x' + y'}{7} = x & & \frac{-x' - 3y'}{7} = y \end{array}$$

Call the two parallel lines $K: ax + by + c = 0$ and $L: ax + by + d = 0$.

$$K': a\left(\frac{2x' + y'}{7}\right) + b\left(-\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' - bx' + 3by' + 7c = 0$$

$$\Rightarrow (2a - b)x' + (a + 3b)y' + 7c = 0$$

$$L': a\left(\frac{2x' + y'}{7}\right) + b\left(-\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' - bx' + 3by' + 7d = 0$$

$$\Rightarrow (2a - b)x' + (a + 3b)y' + 7d = 0$$

$$\therefore K \parallel L \Leftrightarrow K' \parallel L'$$

3 (c) (ii)

$$f(c) = (-1, 9) = (x', y')$$

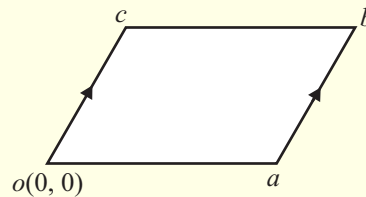
$$x = \frac{2x' + y'}{7} = \frac{2(-1) + (9)}{7} = 1$$

$$y = -\frac{x' - 3y'}{7} = -\frac{(-1) - 3(9)}{7} = 4$$

Therefore, the point $c(1, 4)$.

$$\text{Slope of } oc = \frac{4 - 0}{1 - 0} = 4$$

Therefore, slope of ab is 4 as $oabc$ is a parallelogram.



2004

3 (c) f is the transformation $(x, y) \rightarrow (x', y')$, where $x' = 2x - y$ and $y' = x + y$. L is the line $y = mx + c$. K is the line through the origin that is perpendicular to L .

(i) Find the equation of $f(L)$ and the equation of $f(K)$.

(ii) Find the values of m for which $f(K) \perp f(L)$. Give your answer in surd form.

SOLUTION

3 (c)

$$L: mx - y + c = 0, \text{ Slope} = +\frac{m}{1}$$

$$K: (0, 0), \text{ Slope} = -\frac{1}{m}$$

$$\text{Equation of } K: x + my + k = 0 \Rightarrow (0) + m(0) + k = 0 \Rightarrow k = 0$$

$$K: x + my = 0$$

3 (c) (i)

$$\begin{array}{rcl} x' = 2x - y & \Leftarrow x' = 2x - y \Rightarrow & x' = 2x - y \\ y' = x + y & \Leftarrow y' = x + y \Rightarrow & -2y' = -2x - 2y \\ \hline \frac{x' + y'}{3} = x & & \frac{-x' - 2y'}{3} = y \end{array}$$

$$f(L): m\left(\frac{x' + y'}{3}\right) - \left(-\frac{x' - 2y'}{3}\right) + c = 0 \Rightarrow mx' + my' + x' - 2y' + 3c = 0$$

$$\Rightarrow (m+1)x' + (m-2)y' + 3c = 0$$

$$f(K): \left(\frac{x' + y'}{3}\right) + m\left(-\frac{x' - 2y'}{3}\right) = 0 \Rightarrow x' + y' - mx' + 2my' = 0$$

$$\Rightarrow (1-m)x' + (2m+1)y' = 0$$

3 (c) (ii)

2. PERPENDICULAR LINES

Two lines are perpendicular if the product of their slopes is -1 .

$$K \perp L \Leftrightarrow m_1 \times m_2 = -1. \text{ If } m_1 = -\frac{a}{b} \Rightarrow m_2 = \frac{b}{a}$$

$$\left(-\frac{1-m}{2m+1}\right)\left(-\frac{m+1}{m-2}\right) = -1 \Rightarrow (1-m)(m+1) = -1(2m+1)(m-2)$$

$$\Rightarrow m^2 - 3m - 1 = 0 \Rightarrow m = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} = \frac{3 \pm \sqrt{9+4}}{2}$$

$$\Rightarrow m = \frac{3 \pm \sqrt{13}}{2}$$

2006

3 (c) (ii) L is the line $y = 4x$ and K is the line $x = 4y$. f is the transformation $(x, y) \rightarrow (x', y')$, where $x' = 2x - y$ and $y' = x + 3y$. Find the measure of the acute angle between $f(L)$ and $f(K)$, correct to the nearest degree.

SOLUTION

$$\begin{array}{lll} 3x' = 6x - 3y & \Leftrightarrow x' = 2x - y \Rightarrow & x' = 2x - y \\ y' = x + 3y & \Leftrightarrow y' = x + 3y \Rightarrow & -2y' = -2x - 6y \\ \hline \frac{3x' + y'}{7} = x & & \hline -\frac{x' - 2y'}{7} = y \end{array}$$

$$L: 4x - y = 0 \Rightarrow f(L): 4\left(\frac{3x' + y'}{7}\right) - \left(-\frac{x' - 2y'}{7}\right) = 0$$

$$\Rightarrow 12x' + 4y' + x' - 2y' = 0 \Rightarrow 13x' + 2y' = 0$$

$$K: x - 4y = 0 \Rightarrow f(K): \left(\frac{3x' + y'}{7}\right) - 4\left(-\frac{x' - 2y'}{7}\right) = 0$$

$$\Rightarrow 3x' + y' + 4x' - 8y' = 0 \Rightarrow 7x' - 7y' = 0 \Rightarrow x' - y' = 0$$

$$f(L): 13x' + 2y' = 0 \Rightarrow m_1 = -\frac{13}{2}$$

$$f(K): x' - y' = 0 \Rightarrow m_2 = 1$$

$$\tan \theta = \pm \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) \dots\dots \mathbf{7}$$

$$\tan \theta = \left| \frac{-\frac{13}{2} - 1}{1 + (-\frac{13}{2})(1)} \right| \Rightarrow \tan \theta = \left| \frac{-13 - 2}{2 - 13} \right| = \left| \frac{-15}{-11} \right| = \frac{15}{11}$$

$$\therefore \theta = \tan^{-1}\left(\frac{15}{11}\right) = 53.7^\circ$$

Ans: 54°