

LINE (Q 3, PAPER 2)

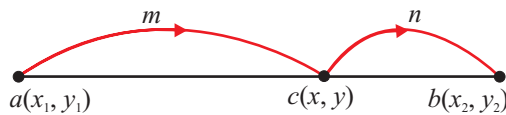
2011

3. (a) P and Q are the points $(-1, 4)$ and $(3, 7)$ respectively.
Find the co-ordinates of the point that divides $[PQ]$ internally in the ratio 3:1.
- (b) f is the transformation $(x, y) \rightarrow (x', y')$, where $x' = x - y$ and $y' = 2x + 3y$.
 l_1 is the line $2x - y - 1 = 0$.
- (i) Find the equation of $f(l_1)$, the image of l_1 under f .
- (ii) Prove that f maps every pair of parallel lines to a pair of parallel lines.
You may assume that f maps every line to a line.
- (iii) The line l_2 is parallel to the line l_1 .
 $f(l_2)$ intersects the x -axis at A' and the y -axis at B' .
The area of the triangle $A'OB'$ is 9 square units, where O is the origin.
Find the two possible equations of l_2 .
- (iv) Given that $A' = f(A)$ and $B' = f(B)$, show that $|\angle AOB| \neq |\angle A'OB'|$.

SOLUTION

3 (a)

INTERNAL DIVISION: The line segment $[ab]$ is divided internally in the ratio $m:n$ to produce point $c(x, y)$.



$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}$$

$P(-1, 4)$ and $Q(3, 7)$

$m:n = 3:1$

Point: $\left(\frac{3(3) + 1(-1)}{3 + 1}, \frac{3(7) + 1(4)}{3 + 1} \right) = \left(\frac{9 - 1}{4}, \frac{21 + 4}{4} \right) = \left(2, \frac{25}{4} \right)$

$$\begin{array}{ccc}
 \mathbf{3 (b)} & 3x' = 3x - 3y & \xleftarrow{\text{red arrow}} \quad x' = x - y \quad \xrightarrow{\text{red arrow}} & -2x' = -2x + 2y \\
 & \frac{y' = 2x + 3y}{3x' + y' = 5x} & & \frac{y' = 2x + 3y}{-2x' + y' = 5y} \\
 & x = \frac{3x' + y'}{5} & & y = \frac{-2x' + y'}{5}
 \end{array}$$

3 (b) (i)

$$l_1: 2x - y - 1 = 0$$

$$f(l_1): 2\left(\frac{3x' + y'}{5}\right) - \left(\frac{-2x' + y'}{5}\right) - 1 = 0$$

$$2(3x' + y') - (-2x' + y') - 5 = 0$$

$$6x' + 2y' + 2x' - y' - 5 = 0$$

$$8x' + y' - 5 = 0$$

3 (b) (ii)

Call the two parallel lines $k: ax + by + c = 0$ and $l: ax + by + d = 0$.

$$k': a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7c = 0$$

$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7c = 0$$

$$l': a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7d = 0$$

$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7d = 0$$

$$\therefore k \parallel l \Leftrightarrow k' \parallel l'$$

3 (b) (iii)

$$l_1: 2x - y - 1 = 0$$

$$l_2: 2x - y + k = 0$$

$$f(l_1): 8x' + y' - 5 = 0$$

$$f(l_2): 8x' + y' + 5k = 0$$

To plot a straight line you need two points. Two good points to choose are where the line cuts the axes.
 Crosses the x -axis: Put $y = 0$ (, 0)
 Crosses the y -axis: Put $x = 0$ (0,)

$$A'\left(-\frac{5k}{8}, 0\right), B'(0, -5k)$$

$$\text{Area} = \frac{1}{2}bh$$

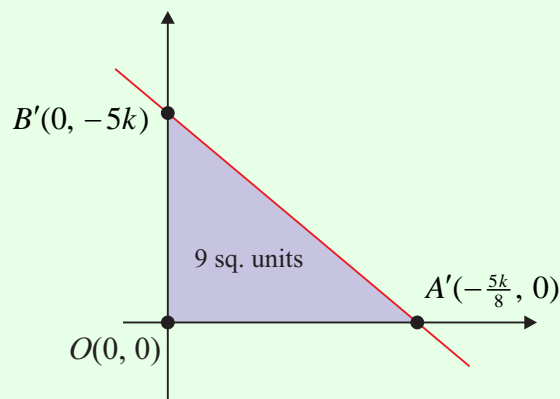
$$\therefore 9 = \frac{1}{2}\left(\frac{5k}{8}\right)(5k)$$

$$144 = 25k^2$$

$$\therefore k = \pm \frac{12}{5}$$

$$f(l_2): 8x' + y' + 5\left(\pm \frac{12}{5}\right) = 0$$

$$8x' + y' \pm 12 = 0$$



3 (b) (iv)

$$A' = f(A) = \left(-\frac{5k}{8}, 0\right) = (x', y')$$

$$x = \frac{3x' + y'}{5} = \frac{3\left(-\frac{5k}{8}\right) + (0)}{5} = \frac{-\frac{15k}{8}}{5} = -\frac{3k}{8}$$

$$y = \frac{-2x' + y'}{5} = \frac{-2\left(-\frac{5k}{8}\right) + (0)}{5} = \frac{\frac{5k}{4}}{5} = \frac{k}{4}$$

$$\therefore A\left(-\frac{3k}{8}, \frac{k}{4}\right)$$

$$B' = f(B) = (0, -5k) = (x', y')$$

$$x = \frac{3x' + y'}{5} = \frac{3(0) + (-5k)}{5} = \frac{-5k}{5} = -k$$

$$y = \frac{-2x' + y'}{5} = \frac{-2(0) + (-5k)}{5} = \frac{-5k}{5} = -k$$

$$\therefore B(-k, -k)$$

As is obvious from the diagram: $|\angle A'OB'| = 90^\circ$

$$A\left(-\frac{3k}{8}, \frac{k}{4}\right), O(0, 0), B(-k, -k)$$

Is $AO \perp BO$?

$$\text{Slope of } AO: m_1 = \frac{\frac{k}{4} - 0}{-\frac{3k}{8} - 0} = \frac{\frac{k}{4}}{-\frac{3k}{8}} = -\frac{2}{3}$$

$$\text{Slope of } BO: m_2 = \frac{0 - (-k)}{0 - (-k)} = \frac{k}{k} = 1$$

Therefore, AO is not perpendicular to BO .

$$|\angle AOB| \neq 90^\circ \Rightarrow |\angle AOB| \neq |\angle A'OB'|$$