INTEGRATION (Q 8, PAPER 1)

2011

8. **(a)** Find $\int (x^3+\sqrt{x})\,dx$.

(b) **(i)** Evaluate $\int_0^2 \frac{x+1}{x^2+2x+2}\,dx$.

(ii) Evaluate $\int_0^2 \frac{x^2+2x+2}{x+1}\,dx$.

(c) Use integration methods to establish the formula $A=\pi r^2$ for the area of a disc of radius r.

SOLUTION

8 (a)

$$I=\int (x^3+\sqrt{x})\,dx=\int (x^3+x^{\frac{1}{2}})\,dx$$

$$=\frac{x^4}{4}+\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c$$

$$=\tfrac{1}{4}x^4+\tfrac{2}{3}x^{\frac{3}{2}}+c$$

$\int x^p\,dx=\frac{x^{p+1}}{p+1}+c$ for $p\in\mathbf{R}$ except $p\neq -1$.

Remember it as:

Add one to the power of *x* and divide by the new power.

8 (b) (i)

STEPS
1. Let *u* equal the *inside* of the more complicated function.
2. Differentiate *u* with respect to *x*.
3. Change the limits from *x* to *u*.
4. Make the substitution.
5. Evaluate the integral *I*.

$$I=\int_0^2 \frac{x+1}{x^2+2x+2}\,dx$$

$$=\tfrac{1}{2}\int_2^{10}\frac{du}{u}$$

$$=\tfrac{1}{2}[\ln u]_2^{10}$$

$$=\tfrac{1}{2}(\ln 10-\ln 2)$$

$$=\tfrac{1}{2}\ln(\tfrac{10}{2})$$

$$=\tfrac{1}{2}\ln 5$$

Make a substitution:

$u=x^2+2x+2$

$du=(2x+2)\,dx$

$du=2(x+1)\,dx$

$\tfrac{1}{2}du=(x+1)\,dx$

Change limits:

$x=2\Rightarrow u=(2)^2+2(2)+2=10$

$x=0\Rightarrow u=(0)^2+2(0)+2=2$

8 (b) (i)

$$I = \int_0^2 \frac{x^2+2x+2}{x+1}\,dx$$

$$= \int_1^3 \frac{(u-1)^2+2(u-1)+2}{u}\,du$$

$$= \int_1^3 \frac{u^2-2u+1+2u-2+2}{u}\,du$$

$$= \int_1^3 \frac{u^2+1}{u}\,du$$

$$= \int_1^3 (u+\tfrac{1}{u})\,du$$

$$= [\tfrac{1}{2}u^2+\ln u]_1^3$$

$$= [(\tfrac{1}{2}(9)+\ln 3)-(\tfrac{1}{2}(1)+\ln 1)]$$

$$= [\tfrac{9}{2}+\ln 3-\tfrac{1}{2}-\ln 1]$$

$$= 4+\ln 3$$

Make a substitution:

$u = x+1 \Rightarrow x = (u-1)$

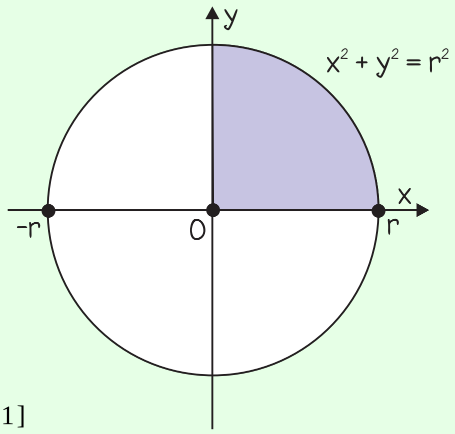
$du = dx$

Change limits:

$x = 2 \Rightarrow u = (2)+1 = 3$

$x = 0 \Rightarrow u = (0)+1 = 1$

8 (c)



$$A = 4\int_0^r y\,dx = 4\int_0^r \sqrt{r^2-x^2}\,dx$$

Let $x = r\sin u$

$dx = r\cos u\,du$

$$A = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2-r^2\sin^2 u}\; r\cos u\,du$$

$$= 4r\int_0^{\frac{\pi}{2}} \sqrt{r^2(1-\sin^2 u)}\,\cos u\,du \quad [\text{Use } \cos^2 u+\sin^2 u = 1]$$

$$= 4r\int_0^{\frac{\pi}{2}} \sqrt{r^2\cos^2 u}\,\cos u\,du$$

$$= 4r\int_0^{\frac{\pi}{2}} r\cos u\cos u\,du$$

$$= 4r^2\int_0^{\frac{\pi}{2}} \cos^2 u\,du$$

$\cos^2 A = \tfrac{1}{2}(1+\cos 2A)$

$$= 4r^2\int_0^{\frac{\pi}{2}} \tfrac{1}{2}(1+\cos 2u)\,du$$

$$= 2r^2\int_0^{\frac{\pi}{2}} (1+\cos 2u)\,du$$

$$= 2r^2[u+\tfrac{1}{2}\sin 2u]_0^{\frac{\pi}{2}}$$

$$= 2r^2[(\tfrac{\pi}{2}+\tfrac{1}{2}\sin 2(\tfrac{\pi}{2}))-(0+\tfrac{1}{2}\sin 2(0))]$$

$$= 2r^2[\tfrac{\pi}{2}+\tfrac{1}{2}\sin\pi-0-\tfrac{1}{2}\sin 0]$$

$$= 2r^2[\tfrac{\pi}{2}+\tfrac{1}{2}(0)-0-\tfrac{1}{2}(0)]$$

$$= 2r^2[\tfrac{\pi}{2}]$$

$$= \pi r^2$$

Change limits:

$x = r:\; r = r\sin u$

$1 = \sin u$

$\therefore u = \sin^{-1} 1 = \tfrac{\pi}{2}$

$x = 0:\; 0 = r\sin u$

$0 = \sin u$

$\therefore u = \sin^{-1} 0 = 0$