

INTEGRATION (Q 8, PAPER 1)

1999

8 (a) Find $\int \left(4x+1+\frac{1}{x^3}\right) dx$.

(b) Evaluate (i) $\int_0^{\frac{\pi}{6}} 2 \cos 4\theta \cos 2\theta d\theta$ (ii) $\int_{-3}^0 (x+3)e^{x(x+6)} dx$.

(c) Evaluate $\int_0^{\sqrt{3}} \sqrt{4-x^2} dx$.

Hint: Let $x = 2 \sin \theta$.

SOLUTION

8 (a)

$$\begin{aligned} \int \left(4x+1+\frac{1}{x^3}\right) dx &= \int (4x+1+x^{-3}) dx \\ &= \frac{4x^2}{2} + x + \frac{x^{-2}}{-2} + c \\ &= 2x^2 + x - \frac{1}{2x^2} + c \end{aligned}$$

$\int x^p dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1$ **1**

Remember it as:

Add one to the power of x and divide by the new power.

8 (b) (i)

$$\begin{aligned} I &= \int_0^{\frac{\pi}{6}} 2 \cos 4\theta \cos 2\theta d\theta \\ \Rightarrow I &= \int_0^{\frac{\pi}{6}} (\cos 6\theta + \cos 2\theta) d\theta \\ \Rightarrow I &= \left[\frac{1}{6} \sin 6\theta + \frac{1}{2} \sin 2\theta\right]_0^{\frac{\pi}{6}} \\ \Rightarrow I &= \left[\left(\frac{1}{6} \sin 6\left(\frac{\pi}{6}\right) + \frac{1}{2} \sin 2\left(\frac{\pi}{6}\right)\right) - \left(\frac{1}{6} \sin 0 + \frac{1}{2} \sin 0\right)\right] \\ \Rightarrow I &= \left[\left(\frac{1}{6} \sin \pi + \frac{1}{2} \sin\left(\frac{\pi}{3}\right)\right) - \left(\frac{1}{6} \sin 0 + \frac{1}{2} \sin 0\right)\right] \\ \Rightarrow I &= \left[\left(0 + \frac{1}{2} \left(\frac{\sqrt{3}}{2}\right)\right) - (0)\right] \\ \therefore I &= \frac{\sqrt{3}}{4} \end{aligned}$$

PRODUCTS \rightarrow SUMS

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

8 (b) (ii)**STEPS**

1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x .
3. Change the limits from x to u .
4. Make the substitution.
5. Evaluate the integral I .

1. Let $u = x^2 + 6x$

2. $du = (2x + 6) dx = 2(x + 3) dx$

$$\Rightarrow \frac{1}{2} du = (x + 3) dx$$

3. $x = 0 \Rightarrow u = (0)^2 + 6(0) = 0$

$$x = -3 \Rightarrow u = (-3)^2 + 6(-3) = 9 - 18 = -9$$

4. $I = \frac{1}{2} \int_{-9}^0 e^u du$ $\int e^x dx = e^x + c$ **3**

5. $I = \frac{1}{2} [e^u]_{-9}^0 = \frac{1}{2} [e^0 - e^{-9}]$

$$\therefore I = \frac{1}{2} \left(1 - \frac{1}{e^9} \right)$$

8 (c)

$$I = \int_0^{\sqrt{3}} \sqrt{4 - x^2} dx$$

1. Let $x = 2 \sin \theta$

2. $dx = 2 \cos \theta d\theta$

3. $x = \sqrt{3} \Rightarrow \sqrt{3} = 2 \sin \theta$

$$\Rightarrow \sin \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = \frac{\pi}{3}$$

$$x = 0 \Rightarrow 0 = 2 \sin \theta$$

$$\Rightarrow \sin \theta = 0 \Rightarrow \theta = \sin^{-1} 0 = 0$$

4. $I = \int_0^{\frac{\pi}{3}} \sqrt{4 - 4 \sin^2 \theta} 2 \cos \theta d\theta$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \sqrt{4(1 - \sin^2 \theta)} 2 \cos \theta d\theta$$
 $\cos^2 A + \sin^2 A = 1$ **8**

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} 2 \sqrt{\cos^2 \theta} 2 \cos \theta d\theta$$

$$\Rightarrow I = 4 \int_0^{\frac{\pi}{3}} \cos^2 \theta d\theta$$
 $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$$\Rightarrow I = \frac{4}{2} \int_0^{\frac{\pi}{3}} (1 + \cos 2\theta) d\theta$$

$$\therefore I = 2 \int_0^{\frac{\pi}{3}} (1 + \cos 2\theta) d\theta$$
 $\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + c$ **5**

$$5. I = 2[\theta + \frac{1}{2} \sin 2\theta]_0^{\frac{\pi}{3}}$$

$$\Rightarrow I = 2[(\frac{\pi}{3} + \frac{1}{2} \sin 2(\frac{\pi}{3})) - (0 + \frac{1}{2} \sin 0)]$$

$$\Rightarrow I = 2[(\frac{\pi}{3} + \frac{1}{2}(\frac{\sqrt{3}}{2})) - (0)]$$

$$\therefore I = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$$