Differentiation & Applications (Q 6 & 7, Paper 1)

Lesson No. 9: Differentiating from First Principles

2006

6 (c) Prove by induction that $\frac{d}{dx}(x^n) = nx^{n-1},\ n \geq 1,\ n \in \mathbf{N}$.

Solution

Statement: If $y = x^n$ prove $\frac{dy}{dx} = nx^{n-1}$ for all $n \in \mathbf{N}_0$.

Proof

Step 1. For $n = 1$: Prove $y = x^1 \Rightarrow \frac{dy}{dx} = 1$

$$y = x$$
$$y + \Delta y = x + \Delta x$$

$$\Delta y = \Delta x$$

$$\therefore \frac{\Delta y}{\Delta x} = 1 \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 1$$

Step 2. $n = k$: Assume $y = x^k \Rightarrow \frac{dy}{dx} = kx^{k-1}$

Step 3. $n = k + 1$: Prove $y = x^{k+1} \Rightarrow \frac{dy}{dx} = (k+1)x^k$

Proof: $y = x^{k+1} = x^k \times x \Rightarrow \frac{dy}{dx} = x^k \times 1 + x \times kx^{k-1}$ (Product Rule)

$$= x^k + kx^k = x^k(k+1)$$

2005

7 (a) Find from first principles the derivative of x^2 with respect to x.

Solution

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$y = x^2$$

$$\Delta y = 2x\Delta x + (\Delta x)^2 \quad \text{by subtraction}$$

$$\therefore \frac{\Delta y}{\Delta x} = 2x + \Delta x \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

2004

6 (b) (ii) Differentiate, from first principles, $\cos x$ with respect to x.

SOLUTION

$$y = f(x) = \cos x$$
$$y + \Delta y = \cos(x + \Delta x)$$
$$y = \cos x$$

$$\therefore \Delta y = \cos(x + \Delta x) - \cos x = -2\sin(x + \tfrac{\Delta x}{2})\sin(\tfrac{\Delta x}{2})$$

$$\therefore \frac{\Delta y}{\Delta x} = -\sin(x + \tfrac{\Delta x}{2})\frac{\sin(\frac{\Delta x}{2})}{(\frac{\Delta x}{2})}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = -\sin x \times 1$$

$$= -\sin x$$

$$\lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1$$ **20**

2002

6 (b) (i) Prove, from first principles, the addition rule:

if $f(x) = u(x) + v(x)$ then $\dfrac{df}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

SOLUTION

1. SUM RULE: If $y = u + v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

PROOF

$$y + \Delta y = (u + \Delta u) + (v + \Delta v)$$
$$y = u + v$$

$$\Delta y = \Delta u + \Delta v$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \Rightarrow \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

2001

6 (b) (ii) Now, find from first principles the derivative of $\sqrt{x}$ with respect to *x*.

SOLUTION

$$y = f(x) = \sqrt{x}$$

$$y + \Delta y = \sqrt{x + \Delta x}$$

$$y = \sqrt{x}$$

$$\therefore \Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{1} \times \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\therefore \Delta y = \frac{x + \Delta x - x}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}$$