

DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

LESSON NO. 9: DIFFERENTIATING FROM FIRST PRINCIPLES

2006

6 (c) Prove by induction that $\frac{d}{dx}(x^n) = nx^{n-1}$, $n \geq 1$, $n \in \mathbf{N}$.

SOLUTION

STATEMENT: If $y = x^n$ prove $\frac{dy}{dx} = nx^{n-1}$ for all $n \in \mathbf{N}_0$.

PROOF

STEP 1. For $n = 1$: Prove $y = x^1 \Rightarrow \frac{dy}{dx} = 1$

$$\begin{array}{l} y = x \\ y + \Delta y = x + \Delta x \\ \hline \Delta y = \Delta x \end{array}$$

$$\therefore \frac{\Delta y}{\Delta x} = 1 \Rightarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 1$$

STEP 2. $n = k$: Assume $y = x^k \Rightarrow \frac{dy}{dx} = kx^{k-1}$

STEP 3. $n = k + 1$: Prove $y = x^{k+1} \Rightarrow \frac{dy}{dx} = (k+1)x^k$

PROOF: $y = x^{k+1} = x^k \times x \Rightarrow \frac{dy}{dx} = x^k \times 1 + x \times kx^{k-1}$ (Product Rule)

$$= x^k + kx^k = x^k(k+1)$$

2005

7 (a) Find from first principles the derivative of x^2 with respect to x .

SOLUTION

$$\begin{array}{l} y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2 \\ y = x^2 \\ \hline \end{array}$$

$$\Delta y = 2x\Delta x + (\Delta x)^2 \text{ by subtraction}$$

$$\therefore \frac{\Delta y}{\Delta x} = 2x + \Delta x \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 2x$$

2004

6 (b) (ii) Differentiate, from first principles, $\cos x$ with respect to x .

SOLUTION

$$y = f(x) = \cos x$$

$$y + \Delta y = \cos(x + \Delta x)$$

$$y = \cos x$$

$$\therefore \Delta y = \cos(x + \Delta x) - \cos x = -2 \sin\left(x + \frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)$$

$$\therefore \frac{\Delta y}{\Delta x} = -\sin\left(x + \frac{\Delta x}{2}\right) \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = -\sin x \times 1$$

$$= -\sin x$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

..... 20

2002

6 (b) (i) Prove, from first principles, the addition rule:

$$\text{if } f(x) = u(x) + v(x) \text{ then } \frac{df}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

SOLUTION

1. SUM RULE: If $y = u + v$ then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

PROOF

$$y + \Delta y = (u + \Delta u) + (v + \Delta v)$$

$$y = u + v$$

$$\Delta y = \Delta u + \Delta v$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \Rightarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

2001

6 (b) (ii) Now, find from first principles the derivative of \sqrt{x} with respect to x .

SOLUTION

$$y = f(x) = \sqrt{x}$$

$$y + \Delta y = \sqrt{x + \Delta x}$$

$$y = \sqrt{x}$$

$$\therefore \Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{1} \times \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$\therefore \Delta y = \frac{x + \Delta x - x}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}$$