DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

LESSON NO. 6: PARAMETRIC DIFFERENTIATION

2006

7 (b) The parametric equations of a curve are:

$$x = 3\cos\theta - \cos^3\theta$$

$$y = 3\sin\theta - \sin^3\theta, \text{ where } 0 < \theta < \tfrac{\pi}{2}.$$

(i) Find $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$.

(ii) Hence show that $\dfrac{dy}{dx} = \dfrac{-1}{\tan^3\theta}$.

SOLUTION

7 (b) (i)

PARAMETRICS: Do $\dfrac{dy}{dt}$ first, then do $\dfrac{dx}{dt}$, and then divide $\dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{dy}{dx}$

$$y = 3\sin\theta - \sin^3\theta \Rightarrow \frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$

$$\Rightarrow \frac{dy}{d\theta} = 3\cos\theta(1-\sin^2\theta) = 3\cos\theta(\cos^2\theta) = 3\cos^3\theta$$

$$x = 3\cos\theta - \cos^3\theta \Rightarrow \frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta$$

$$\Rightarrow \frac{dx}{d\theta} = -3\sin\theta(1-\cos^2\theta) = -3\sin\theta(\sin^2\theta) = -3\sin^3\theta$$

7 (b) (ii)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos^3\theta}{-3\sin^3\theta} = -\frac{1}{\tan^3\theta}$$

2005

7 (b) (i) The parametric equations of a curve are:

$$x = 8 + \ln t^2$$

$$y = \ln(2 + t^2), \text{ where } t > 0.$$

Find $\frac{dy}{dx}$ in terms of t and calculate its value at $t = \sqrt{2}$.

SOLUTION

> **PARAMETRICS**: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$x = 8 + \ln t^2 \Rightarrow \frac{dx}{dt} = \frac{2t}{t^2} = \frac{2}{t}$$

$$y = \ln(2 + t^2) \Rightarrow \frac{dy}{dt} = \frac{2t}{2 + t^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2t}{2+t^2}}{\frac{2}{t}} = \frac{2t}{2+t^2} \times \frac{t}{2} = \frac{t^2}{2+t^2}$$

$$\left(\frac{dy}{dx}\right)_{t=\sqrt{2}} = \frac{(\sqrt{2})^2}{2 + (\sqrt{2})^2} = \frac{2}{2+2} = \frac{1}{2}$$

2004

7 (b) The parametric equations of a curve are:

$$x = 2\theta - \sin 2\theta$$
$$y = 1 - \cos 2\theta, \text{ where } 0 < \theta < \pi.$$

(i) Find $\dfrac{dy}{dx}$.

(ii) Show that the tangent to the curve at $\theta = \frac{\pi}{6}$ is perpendicular to the tangent at $\theta = \frac{2\pi}{3}$.

SOLUTION

7 (b) (i)

PARAMETRICS: Do $\dfrac{dy}{dt}$ first, then do $\dfrac{dx}{dt}$, and then divide $\dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{dy}{dx}$

$$y = 1 - \cos 2\theta \Rightarrow \frac{dy}{d\theta} = 2\sin 2\theta$$

$$x = 2\theta - \sin 2\theta \Rightarrow \frac{dx}{d\theta} = 2 - 2\cos 2\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin 2\theta}{2 - 2\cos 2\theta} = \frac{\sin 2\theta}{1 - \cos 2\theta}$$

7 (b) (ii)

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{6}} = \frac{\sin 2(\frac{\pi}{6})}{1 - \cos 2(\frac{\pi}{6})} = \frac{\sin(\frac{\pi}{3})}{1 - \cos(\frac{\pi}{3})} = \frac{\sin 60^{\circ}}{1 - \cos 60^{\circ}} = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{2\pi}{3}} = \frac{\sin 2(\frac{2\pi}{3})}{1 - \cos 2(\frac{2\pi}{3})} = \frac{\sin(\frac{4\pi}{3})}{1 - \cos(\frac{4\pi}{3})} = \frac{\sin 240^{\circ}}{1 - \cos 240^{\circ}} = \frac{-\sin 60^{\circ}}{1 + \cos 60^{\circ}} = \frac{-\frac{\sqrt{3}}{2}}{1 + \frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = -\frac{1}{\sqrt{3}}$$

$\sqrt{3}\left(-\dfrac{1}{\sqrt{3}}\right) = -1$ [Therefore, the tangents are perpendicular.]

2003

7 (b) (i) The parametric equations of a curve are:

$$x = \cos t + t \sin t$$
$$y = \sin t - t\cos t \text{ where } 0 < t < \tfrac{\pi}{2}.$$

Find $\frac{dy}{dx}$ and write your answer in its simplest form.

SOLUTION

PARAMETRICS: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$y = \sin t - t\cos t \Rightarrow \frac{dy}{dt} = \cos t - \{-t\sin t + \cos t\} = t\sin t$$

$$x = \cos t + t\sin t \Rightarrow \frac{dx}{dt} = -\sin t + \{t\cos t + \sin t\} = t\cos t$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t\sin t}{t\cos t} = \tan t$$

2002

7 (b) (ii) The parametric equations of a curve are $x = \ln(1+t^2)$ and $y = \ln 2t$, where $t \in \mathbf{R}$, $t > 0$. Find the value of $\frac{dy}{dx}$ when $t = \sqrt{5}$.

SOLUTION

PARAMETRICS: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$y = \ln 2t \Rightarrow \frac{dy}{dt} = \frac{2}{2t} = \frac{1}{t}$$

$$x = \ln(1+t^2) \Rightarrow \frac{dx}{dt} = \frac{2t}{(1+t^2)}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t}}{\frac{2t}{(1+t^2)}} = \frac{1}{t} \times \frac{(1+t^2)}{2t} = \frac{(1+t^2)}{2t^2}$$

$$\left(\frac{dy}{dx}\right)_{t=\sqrt{5}} = \frac{(1+(\sqrt{5})^2)}{2(\sqrt{5})^2} = \frac{1+5}{10} = \frac{3}{5}$$

2001

6 (c) Let $x = t^2 e^t$ and $y = t + 2\ln t$ for $t > 0$.

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

(ii) Hence, show that $\frac{dy}{dx} = \frac{1}{x}$.

Solution

6 (c) (i)

Parametrics: Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$x = t^2 e^t \Rightarrow \frac{dx}{dt} = t^2 e^t + e^t(2t) = te^t(t+2)$$

$$y = t + 2\ln t \Rightarrow \frac{dy}{dt} = 1 + \frac{2}{t}$$

6 (c) (ii)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{2}{t}}{te^t(t+2)} \times \frac{t}{t} = \frac{t+2}{t^2 e^t (t+2)} = \frac{1}{t^2 e^t} = \frac{1}{x}$$