DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

LESSON NO. 2: TRIGONOMETRIC DIFFERENTIATION

2005

6 (b) Let $y = \dfrac{1-\cos x}{1+\cos x}$.

Show that $\dfrac{dy}{dx} = t + t^3$, where $t = \tan\left(\dfrac{x}{2}\right)$.

SOLUTION

THE QUOTIENT RULE: If $y = \dfrac{u}{v}$ then:

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad \ldots\ldots\ \textbf{4}$$

Use the quotient rule: $y = \dfrac{1-\cos x}{1+\cos x}$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\sin x + \cos x \sin x + \sin x - \cos x \sin x}{(1+\cos x)^2} = \frac{2\sin x}{(1+\cos x)^2}$$

The second part of this question involves proving a trig identity.

You are required to prove that $\dfrac{dy}{dx} = t + t^3 \Rightarrow \dfrac{2\sin x}{(1+\cos x)^2} = \tan\left(\dfrac{x}{2}\right) + \tan^3\left(\dfrac{x}{2}\right)$

Deal with half angles: Let $A = \frac{x}{2} \Rightarrow x = 2A$

LHS

$$\frac{2\sin 2A}{(1+\cos 2A)^2} = \frac{4\sin A\cos A}{(1+\cos^2 A - \sin^2 A)^2}$$

$$= \frac{4\sin A\cos A}{(2\cos^2 A)^2} = \frac{4\sin A\cos A}{4\cos^4 A}$$

$$= \frac{\sin A}{\cos^3 A}$$

RHS

$$\tan A + \tan^3 A = \tan A(1+\tan^2 A)$$

$$= \tan A(\sec^2 A) = \frac{\sin A}{\cos A} \times \frac{1}{\cos^2 A}$$

$$= \frac{\sin A}{\cos^3 A}$$

2003

7 (a) Differentiate each of the following with respect to x:

(i) $\cos^4 x$

7 (c) (ii) Given that $f(\theta)=\sin(\theta+\pi)\cos(\theta-\pi)$, find the derivative of $f(\theta)$ and express it in the form $\cos n\theta$ where $n\in\mathbf{Z}$.

SOLUTION

7 (a) (i)

$$y=\cos^4 x=(\cos x)^4 \Rightarrow \frac{dy}{dx}=4(\cos x)^3(-\sin x)=-4\cos^3 x\sin x$$

7 (c) (ii)

$$f(\theta)=\sin(\theta+\pi)\cos(\theta-\pi)=\sin(\theta+180^\circ)\cos(\theta-180^\circ)$$

$\sin(\theta+180^\circ)=-\sin\theta$ and $\cos(\theta-180^\circ)=-\cos\theta$ [You can work these out by expanding compound angles using the formulae on page 9 or by using ASTC and recognising them as well-behaved angles.]

$$\Rightarrow f(\theta)=(-\sin\theta)(-\cos\theta)=\sin\theta\cos\theta=\tfrac{1}{2}\sin 2\theta$$

$$\therefore f'(\theta)=\tfrac{1}{2}(2\cos 2\theta)=\cos 2\theta$$

2002

6 (b) (ii) Given $y=2x-\sin 2x$, find $\frac{dy}{dx}$. Give your answer in the form $k\sin^2 x$, where $k\in\mathbf{Z}$.

SOLUTION

$$y=2x-\sin 2x \Rightarrow \frac{dy}{dx}=2-2\cos 2x=2(1-\cos 2x)$$

$$\Rightarrow \frac{dy}{dx}=2(1-\cos^2 x+\sin^2 x)=2(2\sin^2 x)=4\sin^2 x$$

2001

7 (b) (ii) Given that $y=\sin x\cos x$, find $\frac{dy}{dx}$ and express it in the form $\cos nx$ where $n\in\mathbf{Z}$.

SOLUTION

$y=\sin x\cos x=\tfrac{1}{2}\sin 2x$ | $\sin 2A=2\sin A\cos A$ | 13

$$\Rightarrow \frac{dy}{dx}=\tfrac{1}{2}(2\cos 2x)=\cos 2x$$