DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

LESSON NO. 13: NEWTON-RHAPSN APPROXIMATION

2006

7 (a) Taking $x_1 = 2$ as the first approximation to the real root of the equation $x^3 + x - 9 = 0$, use the Newton-Raphson method to find x_2, the second approximation.

SOLUTION

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots\ldots \; \mathbf{16}$$

For $n = 1$: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

For $n = 2$: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

STEPS
1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$$f(x) = x^3 + x - 9 \Rightarrow f'(x) = 3x^2 + 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^3 + 2 - 9}{3(2)^2 + 1} = 2 - \frac{1}{13} = \frac{25}{13}$$

2005

7 (c) (i) Write down a quadratic equation whose roots are $\pm\sqrt{k}$.

(ii) Hence use the Newton-Raphson method to show that the rule $u_{n+1} = \frac{(u_n)^2 + k}{2u_n}$ can be used to find increasingly accurate approximations for $\sqrt{k}$.

(iii) Using the above rule and taking $\frac{3}{2}$ as the first approximation for $\sqrt{3}$, find the third approximation, as a fraction.

SOLUTION

7 (c) (i)

Roots: $\sqrt{k}, -\sqrt{k}$

Quadratic Equation: $(x - \sqrt{k})(x + \sqrt{k}) = 0 \Rightarrow x^2 - k = 0$

7 (c) (ii)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots\ldots \; \mathbf{16}$$

For $n = 1$: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

For $n = 2$: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

STEPS
1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$f(x) = x^2 - k \Rightarrow f'(x) = 2x$

$f(u_n) = (u_n)^2 - k$ and $f'(u_n) = 2u_n$

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)} = u_n - \frac{(u_n)^2 - k}{2u_n} = \frac{2(u_n)^2 - (u_n)^2 + k}{2u_n} = \frac{(u_n)^2 + k}{2u_n}$$

7 (c) (iii)

$u_1 = \frac{3}{2}$, $k = 3$

$$u_2 = \frac{(u_1)^2 + 3}{2u_1} = \frac{(\frac{3}{2})^2 + 3}{2(\frac{3}{2})} = \frac{\frac{9}{4} + 3}{3} = \frac{\frac{21}{4}}{3} = \frac{7}{4}$$

$$u_3 = \frac{(u_2)^2 + 3}{2u_2} = \frac{(\frac{7}{4})^2 + 3}{2(\frac{7}{4})} = \frac{\frac{49}{16} + 3}{\frac{7}{2}} \times \frac{16}{16} = \frac{49 + 48}{56} = \frac{97}{56}$$

2003

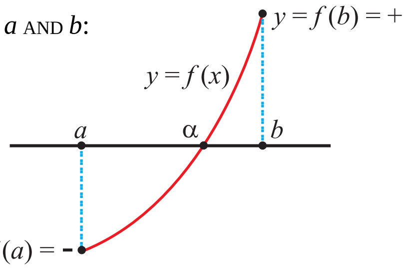
6 (b) Show that the equation $x^3 - 4x - 2 = 0$ has a root between 2 and 3.
Taking $x_1 = 2$ as the first approximation to this root, use the Newton-Raphson method to find x_3, the third approximation. Give your answer correct to two decimal places.

SOLUTION

TEST FOR A ROOT, α, BETWEEN INTEGERS a AND b:

1. Substitute in a: $f(a)$
2. Substitute in b: $f(b)$

If the sign of $f(a)$ and $f(b)$ are different, there is a root between a and b.



$f(2) = (2)^3 - 4(2) - 2 = 8 - 8 - 2 = -2 < 0$

$f(3) = (3)^3 - 4(3) - 2 = 27 - 12 - 2 = 13 > 0$

Therefore, there is a root between 2 and 3.

$f(x) = x^3 - 4x - 2 \Rightarrow f'(x) = 3x^2 - 4$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(x_1)} = 2 - \frac{-2}{3(2)^2 - 4} = 2 - \frac{-2}{8} = 2 \cdot 25$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2 \cdot 25 - \frac{f(2 \cdot 25)}{f'(2 \cdot 25)} = 2 \cdot 25 - \frac{(2 \cdot 25)^3 - 4(2 \cdot 25) - 2}{3(2 \cdot 25)^2 - 4} = 2 \cdot 22$$

2001

7 (a) Taking $x_1 = 1$ as the first approximation to the real root of the equation $x^3 + x^2 - 1 = 0$, use the Newton-Rhapson method to find x_2, the second approximation.

SOLUTION

7 (a)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{.......} \ \mathbf{16}$$

For $n = 1$: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

For $n = 2$: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$.

STEPS

1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

$$f(x) = x^3 + x^2 - 1 = 0 \Rightarrow f'(x) = 3x^2 + 2x$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^3 + 1^2 - 1}{3(1)^2 + 2(1)} = 1 - \frac{1}{5} = \frac{4}{5} = 0 \cdot 8$$