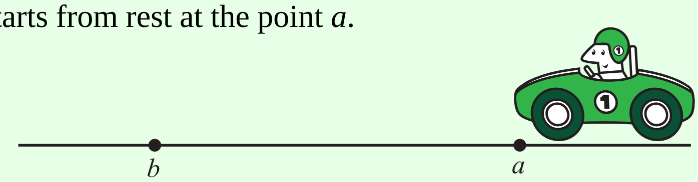
DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

LESSON NO. 9: RATES OF CHANGE

2007

7 (c) A car starts from rest at the point *a*.



The distance of the car from *a*, after *t* seconds, is given by

$$s = 2t^2 + 2t$$

where *s* is in metres.

(i) Find the speed of the car after 2 seconds.

(ii) Find the acceleration of the car.

(iii) The distance from *a* to the point *b* is 24 metres. After how many seconds does the car reach the point *b*?

SOLUTION

7 (c) (i)

Draw up an *s*, *v*, *a* table as shown on the right.

$$s = 2t^2 + 2t$$

$$v = \frac{ds}{dt} = 2\times 2t + 2 = 4t + 2$$

$$a = \frac{dv}{dt} = 4$$

Put *t* = 2 s into the velocity equation.

$v = \frac{ds}{dt}$ **8**

$$v = 4t + 2$$

$$\Rightarrow v = 4(2) + 2 = 8 + 2 = 10 \text{ m/s}$$

$a = \frac{dv}{dt}$ **9**

7 (c) (ii)

$a = 4$ m/s^2

7 (c) (iii)

Put the distance equation equal to 24 m and solve for *t*.

$$s = 24 \Rightarrow 2t^2 + 2t = 24$$

$$\Rightarrow 2t^2 + 2t - 24 = 0$$

$$\Rightarrow t^2 + t - 12 = 0$$

$$\Rightarrow (t+4)(t-3) = 0$$

$$\Rightarrow t = -4, 3$$

Ignore the negative solution. Therefore *t* = 3 s.

2006

7 (c) A missile is fired straight up in the air. The height, h metres, of the missile above the firing position is given by

$$h = t(200 - 5t)$$

where t is the time in seconds from the instant the missile was fired.

(i) Find the speed of the missile after 10 seconds.

(ii) Find the acceleration of the missile.

(iii) One second before reaching its greatest possible height, the missile strikes a target. Find the height of the target.

SOLUTION

7 (c) (i)

Draw up a s, v, a table as shown on the right.

$v = 200 - 10t = 200 - 10(10)$

$= 200 - 100 = 100$ m/s

$v = \frac{ds}{dt}$ **8**

$a = \frac{dv}{dt}$ **9**

$h = t(200 - 5t) = 200t - 5t^2$

$v = \frac{dh}{dt} = 200 - 25t$

$a = \frac{dv}{dt} = -25$

7 (c) (ii)

$a = -10$ m/s^2

7 (c) (iii)

At its greatest possible height, its velocity is zero.
Put $v = 0$ m/s and solve for t.

$v = 200 - 10t \Rightarrow 0 = 200 - 10t$

$\Rightarrow 0 = 20 - t \Rightarrow t = 20$ s

One second before reaching its greatest height ($t = 19$ s), it hits a target.

Find the height after 19 s.

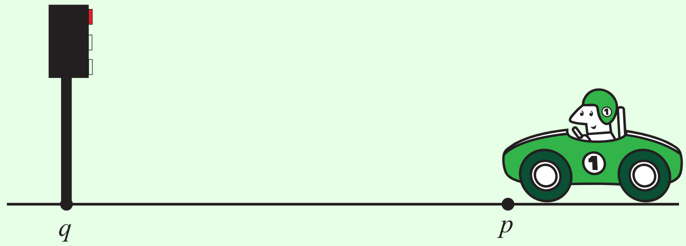
$\therefore h = t(200 - 5t) = (19)(200 - 5(19))$

$\Rightarrow h = (19)(200 - 95)$

$\Rightarrow h = (19)(105) = 1995$ m

2005

7 (c) A car begins to slow down at p in order to stop at a red traffic light at q.



The distance of the car from p, after t seconds, is given by

$$s = 12t - \tfrac{3}{2}t^2$$

where s is in metres.

(i) Find the speed of the car as it passes p.

(ii) Find the time taken to stop.

(iii) The car stops exactly at q. Find the distance from p to q.

SOLUTION

7 (c) (i)

Draw up a s, v, a table as shown on the right.
You are asked to calculate the speed v at t = 0 seconds.

$$v = 12 - 3t = 12 - 3(0) = 12 \text{ m/s}$$

$v = \dfrac{ds}{dt}$ 8

$a = \dfrac{dv}{dt}$ 9

$$s = 12t - \tfrac{3}{2}t^2$$

$$v = \frac{ds}{dt} = 12 - \tfrac{3}{2} \times 2t = 12 - 3t$$

$$a = \frac{dv}{dt} = 0 - 3 = -3$$

7 (c) (ii)

When the car stops its speed v is zero. Put the speed equation equal to zero and solve for t.

$$v = 12 - 3t \Rightarrow 0 = 12 - 3t$$

$$\Rightarrow 3t = 12$$

$$\therefore t = 4 \text{ s}$$

7 (c) (iii)

It takes 4 s to stop. Put t = 4 in the distance s equation.

$$s = 12t - \tfrac{3}{2}t^2$$

$$\Rightarrow s = 12(4) - \tfrac{3}{2}(4)^2$$

$$\Rightarrow s = 48 - \tfrac{3}{2}(16)$$

$$\Rightarrow s = 48 - 24$$

$$\therefore s = 24 \text{ m}$$

2004

7 (c) A jet is moving along an airport runway. At the instant it passes a marker it begins to accelerate for take-off. From the time the jet passes the marker, its distance from the marker is given by

$$s = 2t^2 + 3t,$$

where s is in metres and t is in seconds.

(i) Find the speed of the jet at the instant it passes the marker ($t = 0$).

(ii) The jet has to reach a speed of 83 metres per second to take off. After how many seconds will the jet reach this speed?

(iii) How far is the jet from the marker at that time?

(iv) Find the acceleration of the jet.

Solution

7 (c) (i)

Draw up a s, v, a table as shown on the right.
You are asked to find the speed v at time $t = 0$.

$v = 4t + 3 = 4(0) + 3 = 3$ m/s

$$s = 2t^2 + 3t$$
$$v = \frac{ds}{dt} = 4t + 3$$
$$a = \frac{dv}{dt} = 4$$

$v = \frac{ds}{dt}$ **8**

$a = \frac{dv}{dt}$ **9**

7 (c) (ii)

You are asked to find the time t it takes to reach a speed v of 83 metres per second.

$v = 4t + 3 \Rightarrow 83 = 4t + 3$

$\Rightarrow 4t = 80 \Rightarrow t = 20$ s

7 (c) (iii)

You are asked to find the distance s travelled after a time t of 20 s.

$s = 2t^2 + 3t \Rightarrow s = 2(20)^2 + 3(20)$

$\Rightarrow s = 800 + 60 = 860$ m

7 (c) (iv)

$a = 4$ m/s^2

2003

7 (c) A train is travelling along a track. Suddenly, the brakes are applied. From the time the brakes are applied (t = 0 seconds), the distance travelled by the train, in metres, is given by

$$s = 30t - \tfrac{1}{4}t^2.$$

(i) What is speed of the train at the moment the brakes are applied?
(ii) How many seconds does it take for the train to come to rest?
(iii) How far does the train travel in that time?

Solution

7 (c) (i)

Draw up an s, v, a table as shown on the right.
You are asked to find the speed v at time t = 0.

$$v = 30 - \tfrac{1}{2}t \Rightarrow v = 30 - \tfrac{1}{2}(0) = 30 \text{ m/s}$$

$v = \dfrac{ds}{dt}$ 8

$a = \dfrac{dv}{dt}$ 9

$$s = 30t - \tfrac{1}{4}t^2$$
$$v = \frac{ds}{dt} = 30 - \tfrac{1}{2}t$$
$$a = \frac{dv}{dt} = -\tfrac{1}{2}$$

7 (c) (ii)

You are asked to find the time t it takes for the train to stop, i.e. v = 0 m/s.

$$v = 30 - \tfrac{1}{2}t \Rightarrow 0 = 30 - \tfrac{1}{2}t$$
$$\Rightarrow \tfrac{1}{2}t = 30 \Rightarrow t = 60 \text{ s}$$

7 (c) (iii)

You are asked to find the distance s travelled after 60 s.

$$s = 30t - \tfrac{1}{4}t^2 \Rightarrow s = 30(60) - \tfrac{1}{4}(60)^2$$
$$= 1800 - \tfrac{1}{4}(3600) = 1800 - 900 = 900 \text{ m/s}$$

2002

7 (c) A marble rolls along the top of a table. It starts to move at $t = 0$ seconds. The distance that it has travelled at t seconds is given by

$$s = 14t - t^2$$

where s is in centimetres.

(i) What distance has the marble travelled when $t = 2$ seconds?
(ii) What is the speed of the marble when $t = 5$ seconds?
(iii) When is the speed of the marble equal to zero?
(iv) What is the acceleration of the marble?

SOLUTION

7 (c) (i)

Draw up a s, v, a table as shown on the right.
you are asked to find the distance s travelled after a time $t = 2$ seconds.

$$v = \frac{ds}{dt} \quad \ldots\ldots \quad 8$$

$$a = \frac{dv}{dt} \quad \ldots\ldots \quad 9$$

$$s = 14t - t^2 \Rightarrow s = 14(2) - (2)^2 = 28 - 4$$

$$\therefore s = 24 \text{ cm}$$

$$s = 14t - t^2$$

$$v = \frac{ds}{dt} = 14 - 2t$$

$$a = \frac{dv}{dt} = -2$$

7 (c) (ii)

You are asked to find the speed v when $t = 5$ seconds.

$$v = 14 - 2t \Rightarrow v = 14 - 2(5) = 14 - 10$$

$$\therefore v = 4 \text{ cm/s}$$

7 (c) (iii)

You are asked to find the time t when the speed v is zero.

$$v = 14 - 2t \Rightarrow 0 = 14 - 2t$$

$$\Rightarrow 2t = 14$$

$$\therefore t = 7 \text{ s}$$

7 (c) (iv)

$$a = -2 \text{ cm/s}^2$$

2001

7 (c) Two fireworks were fired straight up in the air at $t = 0$ seconds.
The height, h metres, which each firework reached above the ground t seconds after it was fired is given by

$$h = 80t - 5t^2.$$

The first firework exploded 5 seconds after it was fired.

(i) At what height was the first firework when it exploded?

(ii) At what speed was the first firework travelling when it exploded?

The second firework failed to explode and it fell back to the ground.

(iii) After how many seconds did the second firework reach its maximum height?

SOLUTION

7 (c) (i)

Draw up a s, v, a table as shown on the right.
The first firework exploded after 5 seconds. You are asked to calculate the height h after a time $t = 5$ s.

$v = \frac{ds}{dt}$ 8

$a = \frac{dv}{dt}$ 9

$$h = 80t - 5t^2$$
$$v = \frac{dh}{dt} = 80 - 10t$$
$$a = \frac{dv}{dt} = -10$$

$h = 80t - 5t^2 \Rightarrow h = 80(5) - 5(5)^2$

$= 400 - 5 \times 25 = 400 - 125$

$\therefore h = 275$ m

7 (c) (ii)

You are asked to calculate the speed v of the first firework after a time $t = 5$ s.

$v = 80 - 10t \Rightarrow v = 80 - 10(5) = 80 - 50$

$\therefore v = 30$ m/s

7 (c) (iii)

The second firework reaches its maximum height when its velocity is zero. You need to find out the time t it takes for its velocity v to be zero.

$v = 80 - 10t \Rightarrow 0 = 80 - 10t$

$\Rightarrow 10t = 80$

$\therefore t = 8$ s

2000

7 (c) A car, starting at $t = 0$ seconds, travels a distance of s metres in t seconds where

$$s = 30t - \tfrac{9}{4}t^2.$$

(i) Find the speed of the car after 2 seconds.
(ii) After how many seconds is the speed of the car equal to zero?
(iii) Find the distance travelled by the car up to the time its speed is zero.

SOLUTION

7 (c) (i)

Draw up a s, v, a table as shown on the right. you are asked to find the speed v after a time $t = 2$ seconds.

$v = 30 - \tfrac{9}{2}t \Rightarrow v = 30 - \tfrac{9}{2}(2) = 30 - 9$

$\therefore v = 21$ m/s

$v = \dfrac{ds}{dt}$ 8

$a = \dfrac{dv}{dt}$ 9

$s = 30t - \tfrac{9}{4}t^2$

$v = \dfrac{ds}{dt} = 30 - \tfrac{9}{4} \times 2t = 30 - \tfrac{9}{2}t$

$a = \dfrac{dv}{dt} = -\tfrac{9}{2}$

7 (c) (ii)

You are asked to find the time t when the speed v is zero.

$v = 30 - \tfrac{9}{2}t \Rightarrow 0 = 30 - \tfrac{9}{2}t$

$\Rightarrow 30 = \tfrac{9}{2}t \Rightarrow 60 = 9t$

$\therefore t = \tfrac{60}{9} = \tfrac{20}{3}$ s

7 (c) (iii)

You are asked to find the distance s travelled after a time $t = \tfrac{20}{3}$ s.

$s = 30t - \tfrac{9}{4}t^2 \Rightarrow s = 30(\tfrac{20}{3}) - \tfrac{9}{4}(\tfrac{20}{3})^2$

$\Rightarrow s = 10(20) - \tfrac{9}{4}(\tfrac{400}{9})$

$\Rightarrow s = 200 - 100$

$\therefore s = 100$ m

1999

7 (c) The speed, v, in metres per second, of a body after t seconds is given by

$$v = 3t(4-t).$$

(i) Find the acceleration at each of the two instants when the speed is 9 metres per second.

(ii) Find the speed at the instant when the acceleration is zero.

SOLUTION

7 (c) (i)

Draw up a v, a table as shown on the right.
Firstly, find the times t when the speed v = 9 m/s.
Then, find the accelerations a at these times.

$v = \dfrac{ds}{dt}$ **8**

$a = \dfrac{dv}{dt}$ **9**

$$v = 3t(4-t) = 12t - 3t^2$$
$$a = \frac{dv}{dt} = 12 - 6t$$

$v = 12t - 3t^2 \Rightarrow 9 = 12t - 3t^2$

$\Rightarrow 3t^2 - 12t + 9 = 0$

$\Rightarrow t^2 - 4t + 3 = 0$

$\Rightarrow (t-1)(t-3) = 0$

$\therefore t = 1 \text{ s}, 3 \text{ s}$

$t = 1 : a = 12 - 6(1) = 12 - 6 = 6 \text{ m/s}^2$

$t = 3 : a = 12 - 6(3) = 12 - 18 = -6 \text{ m/s}^2$

7 (c) (ii)

Firstly, find the time t at which the acceleration is zero. Then, find the speed v at this time.

$a = 12 - 6t \Rightarrow 0 = 12 - 6t$

$\Rightarrow 6t = 12$

$\therefore t = 2 \text{ s}$

$t = 2 : v = 3t(4-t) \Rightarrow v = 3(2)(4-2)$

$\therefore v = 6(2) = 12 \text{ m/s}$

1998

7 (c) The volume of water, V, in cm^3, that remains in a leaking tank after t seconds is given by

$$V = 45000 - 300t + 0.5t^2.$$

(i) After how many seconds will the tank be empty?

(ii) Find the rate of change of the volume with respect to t when $t = 50$ seconds.

SOLUTION

7 (c) (i)

After what time t with the volume $V = 0$?

$V = 45000 - 300t + 0.5t^2 \Rightarrow 0 = 45000 - 300t + 0.5t^2$

$\Rightarrow t^2 - 600t + 90000 = 0$

$\Rightarrow (t-300)(t-300) = 0$

$\therefore t = 300$ s

7 (c) (ii)

You need to find the rate of change of the volume, $\frac{dV}{dt}$, at a time $t = 50$ s, $\left(\frac{dV}{dt}\right)_{t=50}$.

$V = 45000 - 300t + 0.5t^2 \Rightarrow \frac{dV}{dt} = -300 + 0.5 \times 2t = -300 + t$

$\therefore \left(\frac{dV}{dt}\right)_{t=50} = -300 + 50 = -250 \text{ cm}^3/\text{s}$

1997

7 (c) The distance s metres of an object from a fixed point at t seconds is given by

$$s = \frac{t+1}{t+3}.$$

(i) At what time is the object 0.75 m from a fixed point?

(ii) What is the speed of the object, in terms of t, at t seconds?

(iii) After how many seconds will the speed of the object be less than 0.02 m/s?

SOLUTION

7 (c) (i)

Find the time t at which the distance $s = 0.75$ m.

$s = \frac{t+1}{t+3} \Rightarrow 0.75 = \frac{t+1}{t+3}$ [Multiply each side by (t + 3).]

$\Rightarrow 0.75(t+3) = t+1$

$\Rightarrow 0.75t + 2.25 = t+1$

$\Rightarrow 2.25 - 1 = t - 0.75t$

$\Rightarrow 1.25 = 0.25t$

$\therefore t = \frac{1.25}{0.25} = 5$ s

CONT...

7 (c) (ii)

You need to differentiate $s = \frac{t+1}{t+3}$ with respect to t to find the speed v. This requires you to use the quotient rule.

$v = \frac{ds}{dt}$ 8

$$u = t+1 \Rightarrow \frac{du}{dt} = 1$$
$$v = t+3 \Rightarrow \frac{dv}{dt} = 1$$

The Quotient Rule: If $y = \frac{u}{v}$ then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ 3

$$s = \frac{t+1}{t+3}$$

$$\Rightarrow \frac{ds}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2} = \frac{(t+3)(1)-(t+1)(1)}{(t+3)^2}$$

$$\Rightarrow \frac{ds}{dt} = \frac{t+3-t-1}{(t+3)^2} = \frac{2}{(t+3)^2}$$

$$\therefore v = \frac{2}{(t+3)^2}$$

7 (c) (iii)

Find out the time t it takes to reach a speed v = 0.02 m/s.

$v = \frac{2}{(t+3)^2} \Rightarrow 0.02 = \frac{2}{(t+3)^2}$ [Multiply across by $(t+3)^2$.]

$\Rightarrow 0.02(t+3)^2 = 2$

$\Rightarrow (t+3)^2 = \frac{2}{0.02} = 100$

$\Rightarrow (t+3) = \pm 10$ [Take the square root of both sides.]

$\therefore t = 7, -13$ [Ignore the negative solution as time must by positive.]

Therefore, t = 7 seconds. After 7 seconds, the speed will be less than 0.02 m/s.

1996

7 (c) A stone is dropped from a height of 80 metres. Its height h metres above the ground after t seconds is given by

$$h = 80 - t^2.$$

Find

(i) its speed after t seconds

(ii) its speed after 2.5 seconds

(iii) the time it takes to fall the first 14.4 metres.

SOLUTION

7 (c) (i)

Draw up a s, v, a table as shown on the right.

$v = -2t$ m/s

$v = \frac{ds}{dt}$ 8

$a = \frac{dv}{dt}$ 9

$$h = 80 - t^2$$

$$v = \frac{dh}{dt} = -2t$$

$$a = \frac{dv}{dt} = -2$$

7 (c) (ii)

Find its speed v after a time t = 2.5 seconds.

$v = -2t = -2 \times 2.5 = -5$ m/s

7 (c) (iii)

Find the time t it takes to fall a height h = 14.4 m.

$h = 80 - t^2 \Rightarrow 14.4 = 80 - t^2$

$\Rightarrow t^2 = 80 - 14.4$

$\Rightarrow t^2 = 65.6$

$\therefore t = \sqrt{65.6} = 8.1$ s