DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

LESSON NO. 8: TURNING POINTS

2005

6 (c) Let $f(x) = x^2 + px + 10,\ x \in \mathbf{R},$ where $p \in \mathbf{Z}$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) The minimum value of $f(x)$ is at $x = 3$. Find the value of p.

(iii) Find the equation of the tangent to $f(x)$ at the point (0, 10).

SOLUTION

6 (c) (i)

$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$ **1**

REMEMBER IT AS:

Multiply down by the power and subtract one from the power.

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$f(x) = x^2 + px + 10$

$\Rightarrow f'(x) = 2x + p \times 1 + 0 = 2x + p$

6 (c) (ii)

To find the turning point which you are told is a minimum, put $f'(x) = 0$.

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ **6**

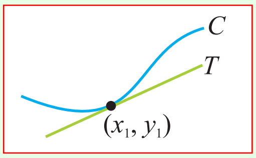
$f'(x) = 0 \Rightarrow 2x + p = 0$

You are told that this minimum is at $x = 3$.

$\therefore 2(3) + p = 0 \Rightarrow 6 + p = 0 \Rightarrow p = -6$

6 (c) (iii)

STEPS TO FINDING THE EQUATION OF A TANGENT, T, AT A POINT (x_1, y_1), ON THE CURVE, C



STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Substitute x_1 in for x to find the slope of the tangent: $\left(\frac{dy}{dx}\right)_{x=x_1}$
3. Find the point of contact (x_1, y_1) by substituting x_1 into the equation of the curve to find y_1.
4. Find the equation of the line of the tangent using formula **4**.

Equation of a line: $y - y_1 = m(x - x_1)$ **4**

(x_1, y_1) is a point on the line and m is the slope of the line.

CONT....

1. $y = f(x) = x^2 - 6x + 10 \Rightarrow \frac{dy}{dx} = 2x - 6$

2. $\left(\frac{dy}{dx}\right)_{x=0} = 2(0) - 6 = -6 \Rightarrow m = -6$

3. Point of contact is (0, 10) = (x_1, y_1).

4. $y - y_1 = m(x - x_1)$

$\Rightarrow y - 10 = -6(x - 0)$

$\Rightarrow y - 10 = -6x$

$\Rightarrow 6x + y - 10 = 0$

2002

6 (c) Let $f(x) = x^3 - ax + 7$ for all $x \in \mathbf{R}$ and for $a \in \mathbf{R}$.

(i) The slope of the tangent to the curve $y = f(x)$ at $x = 1$ is -9.
Find the value of a.

(ii) Hence, find the co-ordinates of the local maximum point and the local minimum point on the curve $y = f(x)$.

SOLUTION

6 (c) (i)

Find the slope of the curve $\frac{dy}{dx}$ at $x = 1$, $\left(\frac{dy}{dx}\right)_{x=1}$, and put it equal to -9.

$y = f(x) = x^3 - ax + 7$

$\therefore \frac{dy}{dx} = 3x^2 - a$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 3(1)^2 - a = -9$

$\Rightarrow 3 - a = -9$

$\therefore a = 12$

6 (c) (ii)

STEPS FOR FINDING THE LOCAL MAXIMUM AND LOCAL MINIMUM OF A FUNCTION:

STEPS

1. Differentiate the function to find $\frac{dy}{dx}$. Differentiate again to find $\frac{d^2y}{dx^2}$.

2. Set $\frac{dy}{dx} = 0$ and solve for x to find the turning points.

3. Substitute the turning points into $\frac{d^2y}{dx^2}$ to decide if they are a local maximum or a local minimum.

4. Find the y coordinates of the turning points by substituting the x values back into the equation of the original function.

CONT....

1. $y = f(x) = x^3 - 12x + 7$

$\frac{dy}{dx} = 3x^2 - 12$

$\frac{d^2y}{dx^2} = 6x$

2. $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12 = 0$

$\Rightarrow 3(x^2 - 4) = 0$

$\Rightarrow 3(x+2)(x-2) = 0$

$\therefore x = -2,\ 2$

3. $\left(\frac{d^2y}{dx^2}\right)_{x=-2} = 6(-2) = -12$

$\left(\frac{d^2y}{dx^2}\right)_{x=2} = 6(2) = 12$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP} < 0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{TP} > 0$

.... **7**

4. $x = -2: y = f(-2) = (-2)^3 - 12(-2) + 7 = -8 + 24 + 7 = 23 \Rightarrow (-2,\ 23)$ is a local maximum.

$x = 2: y = f(2) = (2)^3 - 12(2) + 7 = 8 - 24 + 7 = 23 \Rightarrow (2,\ -9)$ is a local minimum.

2000

8 (c) Let $f(x) = x^3 - 3x^2 + ax + 1$ for all $x \in \mathbf{R}$ and for $a \in \mathbf{R}$.

$f(x)$ has a turning point (a local maximum or a local minimum) at $x = -1$.

(i) Find the value of a.

(ii) Is this turning point a local maximum or a local minimum? Give a reason for your answer.

(iii) Find the co-ordinates of the other turning point of $f(x)$.

SOLUTION

8 (c) (i)

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ **6**

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

You are told there is a turning point at $x = -1$. Therefore, you need to find $\frac{dy}{dx}$ at $x = -1$ and set the answer equal to zero as it is a turning point.

$y = f(x) = x^3 - 3x^2 + ax + 1$

$\therefore \frac{dy}{dx} = 3x^2 - 6x + a$

$\therefore \left(\frac{dy}{dx}\right)_{x=-1} = 3(-1)^2 - 6(-1) + a = 0$

$\Rightarrow 3 + 6 + a = 0$

$\therefore a = -9$

CONT....

8 (c) (ii)

Put the turning point, $x=-1$, into $\frac{d^2y}{dx^2}$.

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP} < 0$
Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{TP} > 0$

.... 7

$y=f(x)=x^3-3x^2-9x+1$

$\Rightarrow \frac{dy}{dx}=3x^2-6x-9$

$\Rightarrow \frac{d^2y}{dx^2}=6x-6$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=-1}=6(-1)-6=-12<0 \Rightarrow$ turning point is a local maximum.

8 (c) (iii) STEPS FOR FINDING THE LOCAL MAXIMUM AND LOCAL MINIMUM OF A FUNCTION:

STEPS

1. Differentiate the function to find $\frac{dy}{dx}$. Differentiate again to find $\frac{d^2y}{dx^2}$.
2. Set $\frac{dy}{dx}=0$ and solve for x to find the turning points.
3. Substitute the turning points into $\frac{d^2y}{dx^2}$ to decide if they are a local maximum or a local minimum.
4. Find the y coordinates of the turning points by substituting the x values back into the equation of the original function.

1. $y=f(x)=x^3-3x^2-9x+1$

$\Rightarrow \frac{dy}{dx}=3x^2-6x-9$

$\Rightarrow \frac{d^2y}{dx^2}=6x-6$

2. $\frac{dy}{dx}=0 \Rightarrow 3x^2-6x-9=0$

$\Rightarrow x^2-2x-3=0$

$\Rightarrow (x-3)(x+1)=0$

$\therefore x=-1, 3$

Anyway, it is the turning point at $x = 3$ which is of interest.

$\therefore y=f(3)=(3)^3-3(3)^2-9(3)+1=27-27-27+1=-26$

Therefore, $(3, -26)$ is the other turning point.

1998

6 (c) $f(x)=(x+k)(x-2)^2$, where k is a real number.

(i) If $f(3) = 7$, find the value of k.

(ii) Using this value for k, find the coordinates of the local maximum and of the local minimum of $f(x)$.

SOLUTION

6 (c) (i)

$f(x)=(x+k)(x-2)^2$

$f(3)=7 \Rightarrow (3+k)(3-2)^2=7$

$\Rightarrow (3+k)(1)^2=7$

$\Rightarrow 3+k=7$

$\therefore k=4$

6 (c) (ii)

$f(x)=(x+4)(x-2)^2$ [Multiply this out and tidy up.]

$\Rightarrow f(x)=(x+4)(x^2-4x+4)$

$\Rightarrow f(x)=x^3-4x^2+4x+4x^2-16x+16$

$\therefore f(x)=x^3-12x+16$

STEPS FOR FINDING THE LOCAL MAXIMUM AND LOCAL MINIMUM OF A FUNCTION:

STEPS

1. Differentiate the function to find $\frac{dy}{dx}$. Differentiate again to find $\frac{d^2y}{dx^2}$.

2. Set $\frac{dy}{dx}=0$ and solve for x to find the turning points.

3. Substitute the turning points into $\frac{d^2y}{dx^2}$ to decide if they are a local maximum or a local minimum.

4. Find the y coordinates of the turning points by substituting the x values back into the equation of the original function.

CONT....

1. $y = f(x) = x^3 - 12x + 16$

$\frac{dy}{dx} = f'(x) = 3x^2 - 12$

$\frac{d^2y}{dx^2} = f''(x) = 6x$

2. $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12 = 0$

$\Rightarrow 3(x^2 - 4) = 0$ $\quad a^2 - b^2 = (a+b)(a-b)$ **1**

$\Rightarrow 3(x+2)(x-2) = 0$

$\therefore x = -2, 2$

3. $\left(\frac{d^2y}{dx^2}\right)_{x=-2} = 6(-2) = -12 < 0$

$\left(\frac{d^2y}{dx^2}\right)_{x=2} = 6(2) = 12 > 0$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP} < 0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{TP} > 0$ **7**

4. $x = -2 : y = f(-2) = (-2)^3 - 12(-2) + 16 = -8 + 24 + 16 = 32 \Rightarrow (-2, 32)$ is a local maximum.

$x = 2 : y = f(2) = (2)^3 - 12(2) + 16 = 8 - 24 + 16 = 0 \Rightarrow (2, 0)$ is a local minimum.

1997

6 (c) Let $f(x) = ax^3 + bx + c$, for all $x \in \mathbf{R}$ and for $a, b, c \in \mathbf{R}$.

Use the information which follows to find the value of a, of b and of c:

(i) $f(0) = 3$

(ii) the slope of the tangent to the curve of $f(x)$ at $x = 1$ is -18

(iii) the curve of $f(x)$ has a local maximum at $x = 2$.

SOLUTION

6 (c) (i)

$f(x) = ax^3 + bx + c$

$f(0) = 3 \Rightarrow a(0)^3 + b(0) + c = 3$

$\therefore c = 3$

6 (c) (ii)

To find the slope of the tangent to the curve at $x = 1$, find $\left(\frac{dy}{dx}\right)_{x=1}$ and put it equal to -18.

$y = f(x) = ax^3 + bx + 3$

$\therefore \frac{dy}{dx} = 3ax^2 + b$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 3a(1)^2 + b = -18$

$\therefore 3a + b = -18$......(**1**)

CONT....

6 (c) (iii)

To find the local maximum you put $\frac{dy}{dx}=0$ and solve for *x*. You are told that you get an answer of *x* = 2 when you do this.

$$\frac{dy}{dx}=0 \Rightarrow 3ax^2+b=0$$
$$\therefore 3a(2)^2+b=0$$
$$\Rightarrow 12a+b=0....(\mathbf{2})$$

You need to solve equations (**1**) and (**2**) to find *a* and *b*.

$$\begin{aligned} 3a+b &= -18..(\mathbf{1})(\times -1) \\ 12a+b &= 0......(\mathbf{2}) \end{aligned} \quad \rightarrow \quad \begin{aligned} -3a-b &= 18 \\ 12a+b &= 0 \\ \hline 9a &= 18 \Rightarrow a=2 \end{aligned}$$

Substitute *a* = 2 into Equation (**2**).

$$\therefore 12(2)+b=0 \Rightarrow 24+b=0$$
$$\therefore b=-24$$

Ans: $a=2,\ b=-24,\ c=3$