Differentiation & Functions (Q 6, 7 & 8, Paper 1)

Lesson No. 7: Tangents

2007

6 (c) Let $f(x) = (5x-2)^4$ for $x \in \mathbf{R}$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) Find the co-ordinates of the point on the curve $y = f(x)$ at which the slope of the tangent is 20.

Solution

6 (c) (i)

$$y = (u)^n \Rightarrow \frac{dy}{dx} = n(u)^{n-1} \times \frac{du}{dx} \quad \ldots 1$$

Remember it as:

Push the power down in front of the bracket and subtract one from the power. Multiply by the differentiation of the inside of the bracket.

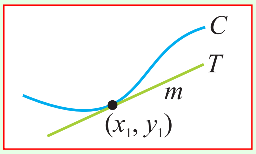
$f(x) = (5x-2)^4$

$\Rightarrow f'(x) = 4(5x-2)^3(5)$

$\Rightarrow f'(x) = 20(5x-2)^3$

6 (c) (ii)

Going backwards: Given the slope of the tangent to the curve, you can work out the point(s) of contact of the tangent with the curve.



Steps

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.
3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

1. $f'(x) = 20(5x-2)^3$

2. $f'(x) = 20 \Rightarrow 20(5x-2)^3 = 20$

$\Rightarrow (5x-2)^3 = 1$

$\Rightarrow 5x-2 = 1$

$\Rightarrow 5x = 3$

$\Rightarrow x = \frac{3}{5}$

3. $f(x) = (5x-2)^4$

$\Rightarrow f(\frac{3}{5}) = (5(\frac{3}{5})-2)^4$

$\Rightarrow f(\frac{3}{5}) = (3-2)^4$

$\Rightarrow f(\frac{3}{5}) = (1)^4 = 1$

Point: $(\frac{3}{5}, 1)$

2003

8 (c) Let $f(x)=x^3+2x^2-1$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) L is the tangent to the curve $y=f(x)$ at $x=-\frac{2}{3}$.
Find the slope of L.

(iii) Find the two values of x at which the tangents to the curve $y=f(x)$ are perpendicular to L.

SOLUTION

8 (c) (i)

$$y=x^n \Rightarrow \frac{dy}{dx}=nx^{n-1} \quad \text{.......} \; ❶$$

REMEMBER IT AS:

Multiply down by the power and subtract one from the power.

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx}=0$

MULTIPLY BY A CONSTANT RULE: If $y=cu$, where c is a constant and u is a function of x, $\frac{dy}{dx}=c\times\frac{du}{dx}$.

$$\left(\frac{dy}{dx}\right)_{x=-\frac{2}{3}}=3(-\tfrac{2}{3})^2+4(-\tfrac{2}{3})=3(\tfrac{4}{9})-\tfrac{8}{3}=\tfrac{4}{3}-\tfrac{8}{3}=-\tfrac{4}{3}$$

8 (c) (ii)

You are being asked to find the slope $\frac{dy}{dx}$ at $x=-\frac{2}{3}$.

$$\left(\frac{dy}{dx}\right)_{x=-\frac{2}{3}}=3(-\tfrac{2}{3})^2+4(-\tfrac{2}{3})=3(\tfrac{4}{9})-\tfrac{8}{3}=\tfrac{4}{3}-\tfrac{8}{3}=-\tfrac{4}{3}$$

8 (c) (iii) Finding the equation of the tangent given its slope:

STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.
3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

L has a slope of $-\frac{4}{3}$. The perpendicular slope is $\frac{3}{4}$.

FINDING THE PERPENDICULAR SLOPE:
Invert the slope and change its sign.

1. $\frac{dy}{dx}=3x^2+4x$

2. $\frac{dy}{dx}=\frac{3}{4}\Rightarrow 3x^2+4x=\frac{3}{4}$ [Multiply across by 4.]

$\Rightarrow 12x^2+16x=3$

$\Rightarrow 12x^2+16x-3=0$

$\Rightarrow (6x-1)(2x+3)=0$

$\therefore x=-\frac{3}{2},\ \frac{1}{6}$

Step **3** is not needed as you are asked to find the y values only.

2000

6 (c) Let $g(x)=(2x+3)(x^2-1)$ for $x \in \mathbf{R}$.

(i) For what two values of x is the slope of the tangent to the curve of $g(x)$ equal to 10?

(ii) Find the equations of the two tangents to the curve of $g(x)$ which have slope 10.

Solution

6 (c) (i)

Steps

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.

2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.

3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

You need to differentiate the function $g(x)$. You can multiply it out and differentiate term by term or you can use the product rule. Here, we multiply it out.

$g(x)=(2x+3)(x^2-1)=2x^3-2x+3x^2-3$

$\Rightarrow g(x)=2x^3+3x^2-2x-3$

$\therefore g'(x)=2\times 3x^2+3\times 2x-2-0$

$\Rightarrow g'(x)=6x^2+6x-2$

1. $g(x)=(2x+3)(x^2-1)$

$g'(x)=6x^2+6x-2$

2. $g'(x)=10 \Rightarrow 6x^2+6x-2=10$

$\Rightarrow 6x^2+6x-12=0$

$\Rightarrow x^2+x-2=0$

$\Rightarrow (x+2)(x-1)=0$

$\therefore x=-2, 1$ [Only the x values are required for part **(i)**. However, the y values are required for part **(ii)**, so continue on to step **3**.]

3. $x=-2: y=g(-2)=(2(-2)+3)((-2)^2-1)=(-1)(3)=-3 \Rightarrow (-2,-3)$ is a point of contact.

$x=1: y=g(1)=(2(1)+3)((1)^2-1)=(5)(0)=0 \Rightarrow (1, 0)$ is a point of contact.

6 (c) (ii)

Equation of a line: $y-y_1=m(x-x_1)$ **4**

(x_1, y_1) is a point on the line and m is the slope of the line.

Tangent 1: Point $(-2, -3)$; $m=10$

$\therefore (y-(-3))=10(x-(-2))$

$\Rightarrow y+3=10(x+2)$

$\Rightarrow y+3=10x+20$

$\Rightarrow 10x-y+17=0$

Tangent 2: Point $(1, 0)$; $m = 10$

$\therefore (y-0)=10(x-1)$

$\Rightarrow y=10x-10$

$\Rightarrow 10x-y-10=0$

1999

6 (c) Let $f(x) = x^3 - 6x^2 + 12$ for $x \in \mathbf{R}$.

Find the derivative of $f(x)$.

At the two points (x_1, y_1) and (x_2, y_2), the tangents to the curve $y = f(x)$ are parallel to the x axis, where $x_2 > x_1$.

Show that

(i) $x_2 - x_1 = 4$

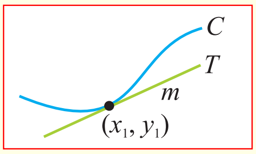
(ii) $y_2 = y_1 - 32$.

Solution

$f(x) = x^3 - 6x^2 + 12$

$\Rightarrow f'(x) = 3x^2 - 12x$

Going backwards: Given the slope of the tangent to the curve, you can work out the point(s) of contact of the tangent with the curve.



Steps

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.
3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

As the tangents are parallel to the x-axis, their slopes are zero.

1. $y = f(x) = x^3 - 6x^2 + 12$

$\Rightarrow \frac{dy}{dx} = f'(x) = 3x^2 - 12x$

2. $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x = 0$

$\Rightarrow 3x(x-4) = 0$

$\therefore x = 0, 4$

3. $x = 0: y = f(0) = (0)^3 - 6(0)^2 + 12 = 12 \Rightarrow (0, 12)$ is a point of contact.

$x = 4: y = f(4) = (4)^3 - 6(4)^2 + 12 = 64 - 96 + 12 \Rightarrow (4, -20)$ is a point of contact.

First point: $(x_1, y_1) = (0, 12)$

Second point: $(x_2, y_2) = (4, -20)$

6 (c) (i)

$x_2 - x_1 = 4 - 0 = 4$ [This is true.]

6 (c) (ii)

$y_2 = y_1 - 32$

$\Rightarrow -20 = 12 - 32$ [This is true.]

1997

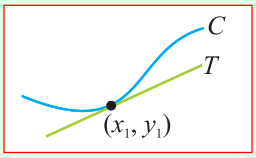
8 (b) Find the equation of the tangent to the curve

$$y = x^3 - 4x + 7$$

at the point where $x = 1$.

SOLUTION

STEPS TO FINDING THE EQUATION OF A TANGENT, T, AT A POINT (x_1, y_1), ON THE CURVE, C



STEPS

1. Differentiate the equation of the curve: $\dfrac{dy}{dx}$.
2. Substitute x_1 in for x to find the slope of the tangent: $\left(\dfrac{dy}{dx}\right)_{x=x_1}$
3. Find the point of contact (x_1, y_1) by substituting x_1 into the equation of the curve to find y_1.
4. Find the equation of the line of the tangent using formula **4**.

Equation of a line: $y - y_1 = m(x - x_1)$ **4**

(x_1, y_1) is a point on the line and m is the slope of the line.

1. $y = f(x) = x^3 - 4x + 7$

$\Rightarrow \dfrac{dy}{dx} = 3x^2 - 4$

2. $\left(\dfrac{dy}{dx}\right)_{x=1} = 3x^2 - 4 = 3(1)^2 - 4 = 3 - 4 = -1 \Rightarrow m = -1$

3. $x = 1: y = f(1) = (1)^3 - 4(1) + 7 = 1 - 4 + 7 = 4 \Rightarrow (x_1, y_1) = (1, 4)$

4. $y - y_1 = m(x - x_1)$

$\Rightarrow y - 4 = -1(x - 1)$

$\Rightarrow y - 4 = -x + 1$

$\Rightarrow x + y - 5 = 0$

1996

6 (c) Let $f(x)=\frac{1}{x-2}$, for $x \in \mathbf{R}$ and $x \neq 2$.

Find the derivative of $f(x)$.
Tangents to $f(x)$ make an angle of 135° with the x axis.
Find the coordinates of the points on the curve of $f(x)$ at which this occurs.

SOLUTION

$$y=f(x)=\frac{1}{x-2}=(x-2)^{-1}$$

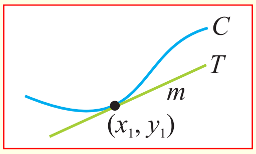
$$\Rightarrow \frac{dy}{dx}=f'(x)=-1(x-2)^{-2}(1)=-\frac{1}{(x-2)^2}$$

You are find the slope by getting the tan of the angle with the x-axis.

$m=\tan 135^{\circ}=-1$ [Using your calculator.]

Slope $m=\tan\theta$

GOING BACKWARDS: Given the slope of the tangent to the curve, you can work out the point(s) of contact of the tangent with the curve.



STEPS

1. Differentiate the equation of the curve: $\frac{dy}{dx}$.

2. Put $\frac{dy}{dx}$ equal to the slope, m, and solve the resulting equation for x to get the x coordinates of the points.

3. Substitute these values of x back into the equation of the curve to get the y coordinates of the points.

1. $y=f(x)=\frac{1}{x-2}$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{(x-2)^2}$$

2. $\frac{dy}{dx}=-1 \Rightarrow -\frac{1}{(x-2)^2}=-1$

$$\Rightarrow \frac{1}{(x-2)^2}=1$$

$$\Rightarrow 1=(x-2)^2$$

$$\Rightarrow \pm 1=x-2$$

$$\therefore x=1,\ 3$$

3. $x=1: y=f(1)=\frac{1}{(1)-2}=\frac{1}{-1}=-1 \Rightarrow (1,-1)$ is a point of contact.

$x=3: y=f(3)=\frac{1}{(3)-2}=\frac{1}{1}=1 \Rightarrow (3,1)$ is a point of contact.