Differentiation & Functions (Q 6, 7 & 8, Paper 1)

Lesson No. 2: Differentiating from First Principles

2007

8 (b) Differentiate $x^2 - 3x$ with respect to x from first principles.

Solution

The Δx approach:

1. $y = x^2 - 3x$

2. $y + \Delta y = (x + \Delta x)^2 - 3(x + \Delta x)$

$\Rightarrow y + \Delta y = x^2 + 2x(\Delta x) + (\Delta x)^2 - 3x - 3(\Delta x)$

3.
$$\begin{array}{rl} y + \Delta y &= x^2 + 2x(\Delta x) + (\Delta x)^2 - 3x - 3(\Delta x) \\ y &= x^2 \qquad\qquad\qquad\qquad\qquad - 3x \\ \hline \therefore \Delta y &= \qquad 2x(\Delta x) + (\Delta x)^2 \qquad - 3(\Delta x) \end{array}$$

4. $\dfrac{\Delta y}{\Delta x} = \dfrac{2x(\Delta x) + (\Delta x)^2 - 3(\Delta x)}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x} = 2x + \Delta x - 3$

6. $\lim\limits_{x \to 0} \dfrac{\Delta y}{\Delta x} = 2x + (0) - 3 = 2x - 3$

7. $\dfrac{dy}{dx} = 2x - 3$

The h approach:

1. $f(x) = x^2 - 3x$

2. $f(x+h) = (x+h)^2 - 3(x+h)$

$\Rightarrow f(x+h) = x^2 + 2xh + h^2 - 3x - 3h$

3. $f(x+h) - f(x)$

$= x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x$

$= 2xh + h^2 - 3h$

4. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2 - 3h}{h}$

5. $\dfrac{f(x+h) - f(x)}{h} = 2x + h - 3$

6. $\lim\limits_{x \to 0} \left(\dfrac{f(x+h) - f(x)}{h} \right) = 2x + (0) - 3$

$= 2x - 3$

7. $\dfrac{dy}{dx} = 2x - 3$

2005

6 (b) Differentiate $3x-x^2$ with respect to x from first principles.

SOLUTION

The Δx approach:

1. $y=3x-x^2$

2. $y+\Delta y=3(x+\Delta x)-(x+\Delta x)^2$

$\Rightarrow y+\Delta y=3x+3(\Delta x)-x^2-2x(\Delta x)-(\Delta x)^2$

3.
$$\begin{aligned} y+\Delta y &= 3x+3(\Delta x)-x^2-2x(\Delta x)-(\Delta x)^2 \\ y &= 3x \qquad\qquad -x^2 \\ \hline \therefore \Delta y &= \qquad 3(\Delta x) \qquad -2x(\Delta x)-(\Delta x)^2 \end{aligned}$$

4. $\dfrac{\Delta y}{\Delta x}=\dfrac{3(\Delta x)-2x(\Delta x)-(\Delta x)^2}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x}=3-2x-\Delta x$

6. $\lim\limits_{x\to 0}\dfrac{\Delta y}{\Delta x}=3-2x-(0)=3-2x$

7. $\dfrac{dy}{dx}=3-2x$

The h approach:

1. $f(x)=3x-x^2$

2. $f(x+h)=3(x+h)-(x+h)^2$

$\Rightarrow f(x+h)=3x+3h-x^2-2xh-h^2$

3. $f(x+h)-f(x)$

$=3x+3h-x^2-2xh-h^2-3x-x^2$

$=3h-2xh-h^2$

4. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{3h-2xh-h^2}{h}$

5. $\dfrac{f(x+h)-f(x)}{h}=3-2x-h$

6. $\lim\limits_{x\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)=3-2x-(0)$

$=3-2x$

7. $\dfrac{dy}{dx}=3-2x$

2004

8 (b) Differentiate x^2+3x with respect to x from first principles.

SOLUTION

The Δx approach:

1. $y = x^2+3x$

2. $y+\Delta y = (x+\Delta x)^2 + 3(x+\Delta x)$

$\Rightarrow y+\Delta y = x^2+2x(\Delta x)+(\Delta x)^2+3x+3(\Delta x)$

3.
$$\begin{aligned} y+\Delta y &= x^2+2x(\Delta x)+(\Delta x)^2+3x+3(\Delta x) \\ y &= x^2 \qquad\qquad\qquad\qquad\quad +3x \\ \hline \therefore \Delta y &= \quad 2x(\Delta x)+(\Delta x)^2 \qquad +3(\Delta x) \end{aligned}$$

4. $\dfrac{\Delta y}{\Delta x} = \dfrac{2x(\Delta x)+(\Delta x)^2+3(\Delta x)}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x} = 2x+\Delta x+3$

6. $\lim\limits_{x\to 0}\dfrac{\Delta y}{\Delta x} = 2x+(0)+3 = 2x+3$

7. $\dfrac{dy}{dx} = 2x+3$

The h approach:

1. $f(x) = x^2+3x$

2. $f(x+h) = (x+h)^2+3(x+h)$

$\Rightarrow f(x+h) = x^2+2xh+h^2+3x+3h$

3. $f(x+h)-f(x)$

$= x^2+2xh+h^2+3x+3h-x^2-3x$

$= 2xh+h^2+3h$

4. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{2xh+h^2+3h}{h}$

5. $\dfrac{f(x+h)-f(x)}{h} = 2x+h+3$

6. $\lim\limits_{x\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right) = 2x+(0)+3$

$= 2x+3$

7. $\dfrac{dy}{dx} = 2x+3$

2003

6 (b) Differentiate $x^2 - 2x$ with respect to x from first principles.

SOLUTION

The Δx approach:

1. $y = x^2 - 2x$

2. $y + \Delta y = (x + \Delta x)^2 - 2(x + \Delta x)$

$\Rightarrow y + \Delta y = x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x)$

3.
$$\begin{array}{rcl} y + \Delta y & = & x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) \\ y & = & x^2 \qquad\qquad\qquad\qquad\quad - 2x \\ \hline \therefore \Delta y & = & \quad 2x(\Delta x) + (\Delta x)^2 \qquad - 2(\Delta x) \end{array}$$

4. $\dfrac{\Delta y}{\Delta x} = \dfrac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x} = 2x + \Delta x - 2$

6. $\lim\limits_{x \to 0} \dfrac{\Delta y}{\Delta x} = 2x + (0) - 2 = 2x - 2$

7. $\dfrac{dy}{dx} = 2x - 2$

The h approach:

1. $f(x) = x^2 - 2x$

2. $f(x+h) = (x+h)^2 - 2(x+h)$

$\Rightarrow f(x+h) = x^2 + 2xh + h^2 - 2x - 2h$

3. $f(x+h) - f(x)$

$= x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x$

$= 2xh + h^2 - 2h$

4. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2 - 2h}{h}$

5. $\dfrac{f(x+h) - f(x)}{h} = 2x + h - 2$

6. $\lim\limits_{x \to 0} \left(\dfrac{f(x+h) - f(x)}{h} \right) = 2x + (0) - 2$

$= 2x - 2$

7. $\dfrac{dy}{dx} = 2x - 2$

2001

8 (b) Differentiate $3x^2 - x$ from first principles with respect to x.

SOLUTION

The Δx approach:

1. $y = 3x^2 - x$

2. $y + \Delta y = 3(x + \Delta x)^2 - (x + \Delta x)$

$\Rightarrow y + \Delta y = 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 - x - (\Delta x)$

3.
$$\begin{array}{rl} y + \Delta y &= 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 - x - (\Delta x) \\ y &= 3x^2 \qquad\qquad\qquad\qquad\quad - x \\ \hline \therefore \Delta y &= \qquad 6x(\Delta x) + 3(\Delta x)^2 \quad - (\Delta x) \end{array}$$

4. $\dfrac{\Delta y}{\Delta x} = \dfrac{6x(\Delta x) + 3(\Delta x)^2 - (\Delta x)}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x} = 6x + 3\Delta x - 1$

6. $\lim\limits_{x \to 0} \dfrac{\Delta y}{\Delta x} = 6x + 3(0) - 1 = 6x - 1$

7. $\dfrac{dy}{dx} = 6x - 1$

The h approach:

1. $f(x) = 3x^2 - x$

2. $f(x+h) = 3(x+h)^2 - (x+h)$

$\Rightarrow f(x+h) = 3x^2 + 6xh + 3h^2 - x - h$

3. $f(x+h) - f(x)$

$= 3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x$

$= 6xh + 3h^2 - h$

4. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{6xh + 3h^2 - h}{h}$

5. $\dfrac{f(x+h) - f(x)}{h} = 6x + 3h - 1$

6. $\lim\limits_{x \to 0} \left(\dfrac{f(x+h) - f(x)}{h} \right) = 6x + 3(0) - 1$

$= 6x - 1$

7. $\dfrac{dy}{dx} = 6x - 1$

2000

6 (a) Differentiate $7x + 3$ from first principles with respect to x.

Solution

The Δx approach:

1. $y = 7x + 3$

2. $y + \Delta y = 7(x + \Delta x) + 3$

$\Rightarrow y + \Delta y = 7x + 7(\Delta x) + 3$

3. $y + \Delta y = 7x + 7(\Delta x) + 3$

$y \quad = 7x \quad\quad + 3$

$\therefore \Delta y = \quad 7(\Delta x)$

4. $\dfrac{\Delta y}{\Delta x} = \dfrac{7(\Delta x)}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x} = 7$

6. $\lim\limits_{x \to 0} \dfrac{\Delta y}{\Delta x} = 7$

7. $\dfrac{dy}{dx} = 7$

The h approach:

1. $f(x) = 7x + 3$

2. $f(x+h) = 7(x+h) + 3$

$\Rightarrow f(x+h) = 7x + 7h + 3$

3. $f(x+h) - f(x)$

$= 7x + 7h + 3 - 7x - 3 = 7h$

4. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{7h}{h}$

5. $\dfrac{f(x+h) - f(x)}{h} = 7$

6. $\lim\limits_{x \to 0} \left(\dfrac{f(x+h) - f(x)}{h} \right) = 7$

7. $\dfrac{dy}{dx} = 7$

1999

6 (b) Differentiate from first principles

$$x^2+5x$$

with respect to x.

SOLUTION

The Δx approach:

1. $y=x^2+5x$

2. $y+\Delta y=(x+\Delta x)^2+5(x+\Delta x)$

$\Rightarrow y+\Delta y=x^2+2x(\Delta x)+(\Delta x)^2+5x+5(\Delta x)$

3. $y+\Delta y=x^2+2x(\Delta x)+(\Delta x)^2+5x+5(\Delta x)$

$y \quad =x^2 \qquad\qquad +5x$

$\therefore \Delta y= \quad 2x(\Delta x)+(\Delta x)^2 \quad +5(\Delta x)$

4. $\dfrac{\Delta y}{\Delta x}=\dfrac{2x(\Delta x)+(\Delta x)^2+5(\Delta x)}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x}=2x+\Delta x+5$

6. $\lim\limits_{x\to 0}\dfrac{\Delta y}{\Delta x}=2x+(0)+5=2x+5$

7. $\dfrac{dy}{dx}=2x+5$

The h approach:

1. $f(x)=x^2+5x$

2. $f(x+h)=(x+h)^2+5(x+h)$

$\Rightarrow f(x+h)=x^2+2xh+h^2+5x+5h$

3. $f(x+h)-f(x)$

$=x^2+2xh+h^2+5x+5h-x^2-5x$

$=2xh+h^2+5h$

4. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{2xh+h^2+5h}{h}$

5. $\dfrac{f(x+h)-f(x)}{h}=2x+h+5$

6. $\lim\limits_{x\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)=2x+(0)+5$

$=2x+5$

7. $\dfrac{dy}{dx}=2x+5$

1997

6 (b) Differentiate from first principles

$$3x^2 - 2$$

with respect to x.

SOLUTION

The Δx approach:

1. $y = 3x^2 - 2$

2. $y + \Delta y = 3(x + \Delta x)^2 - 2$

$\Rightarrow y + \Delta y = 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 - 2$

3. $y + \Delta y = 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 - 2$

$y \quad = 3x^2 \qquad\qquad - 2$

$\therefore \Delta y = 6x(\Delta x) + 3(\Delta x)^2$

4. $\dfrac{\Delta y}{\Delta x} = \dfrac{6x(\Delta x) + 3(\Delta x)^2}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x} = 6x + 3\Delta x$

6. $\lim\limits_{x \to 0} \dfrac{\Delta y}{\Delta x} = 6x + 3(0) = 6x$

7. $\dfrac{dy}{dx} = 6x$

The h approach:

1. $f(x) = 3x^2 - 2$

2. $f(x+h) = 3(x+h)^2 - 2$

$\Rightarrow f(x+h) = 3x^2 + 6xh + 3h^2 - 2$

3. $f(x+h) - f(x)$

$= 3x^2 + 6xh + 3h^2 - 2 - 3x^2 + 2$

$= 6xh + 3h^2$

4. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{6xh + 3h^2}{h}$

5. $\dfrac{f(x+h) - f(x)}{h} = 6x + 3h$

6. $\lim\limits_{x \to 0} \left(\dfrac{f(x+h) - f(x)}{h} \right) = 6x + 3(0)$

$= 6x$

7. $\dfrac{dy}{dx} = 6x$

1996

7 (a) Differentiate from first principles

$$3x-7$$

with respect to x.

SOLUTION

The Δx approach:

1. $y=3x-7$

2. $y+\Delta y=3(x+\Delta x)-7$

$\Rightarrow y+\Delta y=3x+3(\Delta x)-7$

3. $y+\Delta y=3x+3(\Delta x)-7$

$y \qquad = 3x \qquad -7$

$\therefore \Delta y = \qquad 3(\Delta x)$

4. $\dfrac{\Delta y}{\Delta x}=\dfrac{3(\Delta x)}{\Delta x}$

5. $\dfrac{\Delta y}{\Delta x}=3$

6. $\lim\limits_{x\to 0}\dfrac{\Delta y}{\Delta x}=3$

7. $\dfrac{dy}{dx}=3$

The h approach:

1. $f(x)=3x-7$

2. $f(x+h)=3(x+h)-7$

$\Rightarrow f(x+h)=3x+3h-7$

3. $f(x+h)-f(x)$

$=3x+3h-7-3x+7=3h$

4. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{3h}{h}$

5. $\dfrac{f(x+h)-f(x)}{h}=3$

6. $\lim\limits_{x\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)=3$

7. $\dfrac{dy}{dx}=3$