Complex Numbers (Q 4, Paper 1)

Lesson No. 6: Equations I

2007

4 (b) Let $z = 5 - 3i$.

(i) Plot z and $-z$ on an Argand diagram.

(ii) Calculate $|z - 1|$.

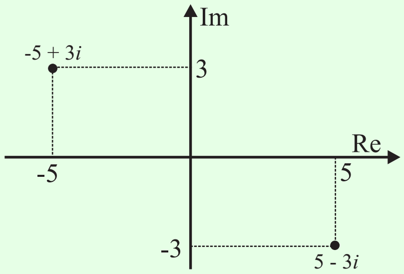
(iii) Find the value of the real number k such that $ki + 4z = 20$.

Solution

4 (b) (i)

$z = 5 - 3i$

$-z = -5 + 3i$



4 (b) (ii)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ 2

$|z - 1| = |5 - 3i - 1| = |4 - 3i|$

$= \sqrt{(4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

4 (b) (iii)

For all equations you can equate (set equal) the real parts and the imaginary parts.

$ki + 4z = 20$

$\Rightarrow ki + 4(5 - 3i) = 20$

$\Rightarrow ki + 20 - 12i = 20$

$\Rightarrow ki - 12i = 0$

$\Rightarrow 0 + (k - 12)i = 0 + 0i$

$\therefore k - 12 = 0 \Rightarrow k = 12$

2006

4 (c) (i) Express $\frac{3-2i}{1-4i}$ in the form $x + yi$.

(ii) Hence, or otherwise, find the values of the real numbers p and q such that

$$p+2qi=\frac{17(3-2i)}{1-4i}.$$

SOLUTION

4 (c) (i)

DIVISION: Multiply above and below by the conjugate of the bottom.

$\frac{3-2i}{1-4i}$ [Multiply above and below by the conjugate of the bottom.]

$=\frac{(3-2i)}{(1-4i)}\times\frac{(1+4i)}{(1+4i)}$ [Multiply out the brackets.]

$=\frac{3+12i-2i-8i^2}{1+4i-4i-16i^2}$ [Tidy up using the fact that $i^2=-1$.]

$=\frac{3+10i+8}{1+16}=\frac{11+10i}{17}$ [Divide the 17 into each term on top.]

$=\frac{11}{17}+\frac{10}{17}i$

4 (c) (ii)

For all equations you can equate (set equal) the real parts and the imaginary parts.

$p+2qi=\frac{17(3-2i)}{1-4i}$

$\Rightarrow p+2qi=\frac{17(11+10i)}{17}$

$\Rightarrow p+2qi=11+10i$ [Equate the real parts and the imaginary parts.]

$\therefore p=11$ and $2q=10\Rightarrow q=5$

2005

4 (c) Let $z = 1 - 2i$.

(i) Write down $\overline{z}$, the complex conjugate of z.

(ii) Find the real numbers k and t such that

$$kz + t\overline{z} = 2z^2.$$

SOLUTION

4 (c) (i)

Working out the conjugate: $z = a + bi \Rightarrow \overline{z} = a - bi$ **1**

$z = 1 - 2i \Rightarrow \overline{z} = 1 + 2i$

4 (c) (ii)

For all equations you can equate (set equal) the real parts and the imaginary parts.

$kz + t\overline{z} = 2z^2 \Rightarrow k(1-2i) + t(1+2i) = 2(1-2i)^2$ [Multiply out the brackets.]

$\Rightarrow k - 2ki + t + 2ti = 2(1 - 4i + 4i^2)$ [Gather up the real and imaginary parts on the left.]

$\Rightarrow (k+t) + (2t - 2k)i = 2(1 - 4i - 4)$

$\Rightarrow (k+t) + (2t - 2k)i = 2(-3 - 4i)$

$\Rightarrow (k+t) + (2t - 2k)i = -6 - 8i$ [Equate the real parts and the imaginary parts.]

Equating the real parts: $k + t = -6$....(**1**)

Equating the imaginary parts: $2t - 2k = -8 \Rightarrow t - k = -4$....(**2**)

Solve Equations (**1**) and (**2**) simultaneously.

$t + k = -6$....(**1**)
$t - k = -4$....(**2**)
$2t \quad = -10 \Rightarrow t = -5$

Substitute this value of t into Eqn. (**1**): $(-5) + k = -6 \Rightarrow k = -6 + 5 \Rightarrow k = -1$

2004

4 (c) Let $z_1 = 5+12i$ and $z_2 = 2-3i$.

(i) Find the value of the real number k such that $|z_1| = k|z_2|$.

(ii) p and q are real numbers such that

$$\frac{z_1}{z_2} = p(q+i).$$

Find the value of p and the value of q.

SOLUTION

4 (c) (i)

Finding the modulus:

$z = a+bi \Rightarrow |z| = \sqrt{a^2+b^2}$ **2**

$|z_1| = k|z_2| \Rightarrow |5+12i| = k|2-3i|$

$\Rightarrow \sqrt{(5)^2+(12)^2} = k\sqrt{(2)^2+(-3)^2}$

$\Rightarrow \sqrt{25+144} = k\sqrt{4+9}$

$\Rightarrow \sqrt{169} = k\sqrt{13}$

$\Rightarrow 13 = k\sqrt{13} \Rightarrow k = \dfrac{13}{\sqrt{13}} = \sqrt{13}$

4 (c) (ii)

DIVISION: Multiply above and below by the conjugate of the bottom.

$\dfrac{z_1}{z_2} = p(q+i) \Rightarrow \dfrac{5+12i}{2-3i} = pq+pi$

$\Rightarrow \dfrac{(5+12i)}{(2-3i)} \times \dfrac{(2+3i)}{(2+3i)} = pq+pi$ [Multiply the left hand side above and below by the conjugate of the bottom.]

$\Rightarrow \dfrac{10+15i+24i+36i^2}{4+6i-6i-9i^2} = pq+pi$ [Tidy up the left hand side using the fact that $i^2 = -1$.]

$\Rightarrow \dfrac{10+39i-36}{4+9} = pq+pi$

$\Rightarrow \dfrac{-26+39i}{13} = pq+pi$

$\Rightarrow -2+3i = pq+pi$ [Equate the real parts and the imaginary parts.]

For all equations you can equate (set equal) the real parts and the imaginary parts.

Equate the imaginary parts: $\therefore p = 3$

Equate the real parts: $\therefore pq = -2 \Rightarrow (3)q = -2 \Rightarrow q = -\frac{2}{3}$

2003

4 (c) Let $w = 1 + i$.

(i) Simplify $\frac{6}{w}$.

(ii) a and b are real numbers such that

$$a\left(\frac{6}{w}\right) - b(w+1) = 3(w+i).$$

Find the value of a and the value of b.

SOLUTION

4 (c) (i)

Working out the conjugate: $z = a + bi \Rightarrow \bar{z} = a - bi$ 1

DIVISION: Multiply above and below by the conjugate of the bottom.

$$\frac{6}{w} = \frac{6}{1+i}$$

$$= \frac{6}{(1+i)} \times \frac{(1-i)}{(1-i)}$$ [Multiply above and below by the conjugate of the bottom.]

$$= \frac{6-6i}{1-i+i-i^2}$$ [Tidy up using the fact that $i^2 = -1$.]

$$= \frac{6-6i}{1+1} = \frac{6-6i}{2}$$

$$= 3 - 3i$$

4 (c) (ii)

$$a\left(\frac{6}{w}\right) - b(w+1) = 3(w+i)$$

$$\Rightarrow a(3-3i) - b(1+i+1) = 3(1+i+i)$$

$$\Rightarrow a(3-3i) - b(2+i) = 3(1+2i)$$

$$\Rightarrow 3a - 3ai - 2b - bi = 3 + 6i$$

$$\Rightarrow 3a - 2b + (-3a - b)i = 3 + 6i$$

For all equations you can equate (set equal) the real parts and the imaginary parts.

Equating the real parts: $3a - 2b = 3 ...(\mathbf{1})$

Equating the imaginary parts: $-3a - b = 6 ...(\mathbf{2})$

Solve the equations (**1**) and (**2**) simultaneously.

$$3a - 2b = 3 ...(\mathbf{1})$$
$$-3a - b = 6 ...(\mathbf{2})(\times -2)$$

→

$$3a - 2b = 3$$
$$6a + 2b = -12$$
$$9a = -9 \Rightarrow a = -1$$

Substitute this value of a into Eqn. (**1**): $3(-1) - 2b = 3 \Rightarrow -3 - 2b = 3 \Rightarrow -2b = 6 \Rightarrow b = -3$

2001

4 (b) Solve

$$(x+2yi)(1-i)=7+5i$$

for real x and for real y.

SOLUTION

For all equations you can equate (set equal) the real parts and the imaginary parts.

$(x+2yi)(1-i)=7+5i$ [Multiply out the brackets.]

$\Rightarrow x-xi+2yi-2yi^2=7+5i$ [Tidy up using the fact that $i^2=-1$.]

$\Rightarrow x-xi+2yi+2y=7+5i$ [Gather up the real parts and the imaginary parts.]

$\Rightarrow (x+2y)+(-x+2y)i=7+5i$ [Equate the real parts and the imaginary parts.]

Equating the real parts: $x+2y=7...(\mathbf{1})$

Equating the imaginary parts: $-x+2y=5....(\mathbf{2})$

Solve equations (**1**) and (**2**) simultaneously.

Substitute this value for y into Eqn. (**1**):

$x+2(3)=7 \Rightarrow x+6=7 \Rightarrow x=1$

$$\begin{aligned} x+2y&=7...(\mathbf{1})\\ -x+2y&=5....(\mathbf{2})\\ \hline 4y&=12 \Rightarrow y=3 \end{aligned}$$

2000

4 (c) Let $z = 2 + 4i$.

(i) Express $z^2 + 28$ in the form $p + qi$ where $p, q \in \mathbf{R}$.

(ii) Solve for real k

$$k(z^2+28)=|z|(1+i).$$

Express your answer in the form $\frac{\sqrt{a}}{b}$ where $a, b \in \mathbf{N}$ and a is a prime number.

SOLUTION

4 (c) (i)

$z^2+28=(2+4i)^2+28$

$=(2+4i)(2+4i)+28$ [Multiply out the brackets.]

$=4+8i+8i+16i^2+28$ [Tidy up using the fact that $i^2=-1$.]

$=32+16i-16$

$=16+16i$

CONT....

4 (c) (ii)

For all equations you can equate (set equal) the real parts and the imaginary parts.

$k(z^2+28)=|z|(1+i)$

$\Rightarrow k(16+16i)=|2+4i|(1+i)$

$\Rightarrow 16k+16ki=\sqrt{2^2+4^2}(1+i)$

$\Rightarrow 16k+16ki=\sqrt{20}(1+i)$

$\Rightarrow 16k+16ki=\sqrt{20}+\sqrt{20}i$

Finding the modulus:

$z=a+bi\Rightarrow |z|=\sqrt{a^2+b^2}$ 2

Equate the real parts: $16k=\sqrt{20}\Rightarrow k=\frac{\sqrt{20}}{16}=\frac{2\sqrt{5}}{16}=\frac{\sqrt{5}}{8}$

1999

4 (c) Let $w=i-2$.

Express w^2 in the form $a+bi,\ \ a, b\in \mathbf{R}$.

Hence, solve

$$kw^2=2w+1+ti$$

for real k and real t.

Solution

$w=i-2\Rightarrow w^2=(i-2)^2=(i-2)(i-2)$ [Multiply out the brackets.]

$=i^2-2i-2i+4$ [Tidy up using the fact that $i^2=-1$.]

$=-1-4i+4$

$=3-4i$

For all equations you can equate (set equal) the real parts and the imaginary parts.

$kw^2=2w+1+ti$

$\Rightarrow k(3-4i)=2(i-2)+1+ti$

$\Rightarrow 3k-4ki=2i-4+1+ti$

$\Rightarrow 3k-4ki=-3+(t+2)i$ [Gather up the real parts and the imaginary parts.]

Equate the real parts: $3k=-3\Rightarrow k=-1$

Equate the imaginary parts: $-4k=t+2\Rightarrow -4(-1)=t+2\Rightarrow 4=t+2\Rightarrow t=2$

1998

4 (c) Let $u = 2 - i$.

(i) Express $u + \frac{1}{u}$ in the form $a + bi, \quad a, b \in \mathbf{R}$.

(ii) Hence, solve

$$k(u + \tfrac{1}{u}) + ti = 18$$

for real k and real t.

SOLUTION

4 (c) (i)

Working out the conjugate: $z = a + bi \Rightarrow \bar{z} = a - bi$ **1**

DIVISION: Multiply above and below by the conjugate of the bottom.

$u + \dfrac{1}{u} = 2 - i + \dfrac{1}{2-i}$

$= 2 - i + \dfrac{1}{(2-i)} \times \dfrac{(2+i)}{(2+i)}$ [Multiply above and below by the conjugate of the bottom.]

$= 2 - i + \dfrac{2+i}{4 + 2i - 2i - i^2}$ [Multiply out the brackets.]

$= 2 - i + \dfrac{2+i}{4+1} = 2 - i + \dfrac{2+i}{5}$ [Tidy up using the fact that $i^2 = -1$.]

$= 2 - i + \frac{2}{5} + \frac{1}{5}i$ [Divide the 5 on the bottom into each term above.]

$= \frac{12}{5} - \frac{4}{5}i$ [Add the real parts and the imaginary parts.]

4 (c) (ii)

For all equations you can equate (set equal) the real parts and the imaginary parts.

$k\left(u + \dfrac{1}{u}\right) + ti = 18$ [Write 18 as a complex number.]

$\Rightarrow k(\frac{12}{5} - \frac{4}{5}i) + ti = 18 + 0i$

$\Rightarrow \frac{12}{5}k - \frac{4}{5}ki + ti = 18 + 0i$

$\Rightarrow \frac{12}{5}k + (-\frac{4}{5}k + t)i = 18 + 0i$ [Gather up the real parts and the imaginary parts.]

Equating the real parts: $\frac{12}{5}k = 18 \Rightarrow k = 18 \times \frac{5}{12} = \frac{15}{2}$

Equating the imaginary parts: $-\frac{4}{5}k + t = 0 \Rightarrow -\frac{4}{5}(\frac{15}{2}) + t = 0 \Rightarrow -6 + t = 0 \Rightarrow t = 6$

1997

4 (c) Let $z = 1 + i$ and let $\bar{z}$ be the complex conjugate of z.

Express $\frac{z}{\bar{z}}$ in the form $x + yi,\ \ x,\ y \in \mathbf{R}$.

Hence solve $k\left(\frac{z}{\bar{z}}\right) + tz = -3 - 4i$

for real k and t.

SOLUTION

Working out the conjugate: $z = a + bi \Rightarrow \bar{z} = a - bi$ 1

DIVISION: Multiply above and below by the conjugate of the bottom.

$z = 1 + i \Rightarrow \bar{z} = 1 - i$

$\frac{z}{\bar{z}} = \frac{1+i}{1-i}$ [Multiply above and below by the conjugate of the bottom.]

$= \frac{(1+i)}{(1-i)} \times \frac{(1+i)}{(1+i)}$ [Multiply out the brackets.]

$= \frac{1+i+i+i^2}{1+i-i-i^2}$ [Tidy up using the fact that $i^2 = -1$.]

$= \frac{1+2i-1}{1+1} = \frac{2i}{2}$

$= 1 = 1 + 0i$

For all equations you can equate (set equal) the real parts and the imaginary parts.

$k\left(\frac{z}{\bar{z}}\right) + tz = -3 - 4i$

$\Rightarrow k(1) + t(1+i) = -3 - 4i$

$\Rightarrow k + t + ti = -3 - 4i$

$\Rightarrow (k+t) + ti = -3 - 4i$ [Gather up the real parts and the imaginary parts.]

Equate the imaginary parts: $t = -4$

Equate the real parts: $k + t = -3 \Rightarrow k - 4 = -3 \Rightarrow k = 1$

1996

4 (b) Let $w=(1-3i)(2+i)$.

Express w in the form $p+qi,\quad p,\ q\in\mathbf{R}$.

Verify that

$$|w+\bar{w}|=|w-\bar{w}|,$$

where $\bar{w}$ is the complex conjugate of w.

For what value of a is

$$\frac{\bar{w}}{2i}=aw,$$

where $a\in\mathbf{R}$?

Solution

$w=(1-3i)(2+i)$ [Multiply out the brackets.]

$=2+i-6i-3i^2$ [Tidy up using the fact that $i^2=-1$.]

$=2+i-6i+3$ [Add the real parts and the imaginary parts.]

$=5-5i$

Working out the conjugate: $z=a+bi\Rightarrow\bar{z}=a-bi$ 1

Finding the modulus: $z=a+bi\Rightarrow|z|=\sqrt{a^2+b^2}$ 2

$w=5-5i\Rightarrow\bar{w}=5+5i$

LHS	*LHS*
$\lvert w+\bar{w}\rvert$	$\lvert w-\bar{w}\rvert$
$=\lvert 5-5i+5+5i\rvert$	$=\lvert 5-5i-5-5i\rvert$
$=\lvert 10+0i\rvert$	$=\lvert 0-10i\rvert$
$=\sqrt{10^2+0^2}$	$=\sqrt{0^2+(-10)^2}=\sqrt{0+100}$
$=\sqrt{100}=10$	$=\sqrt{100}=10$

For all equations you can equate (set equal) the real parts and the imaginary parts.

$$\frac{\bar{w}}{2i}=aw$$

$$\Rightarrow\frac{5+5i}{2i}=a(5-5i)$$

$$\Rightarrow\frac{(5+5i)}{2i}\times\frac{i}{i}=5a-5ai$$

$$\Rightarrow\frac{5i+5i^2}{2i^2}=5a-5ai$$ [Tidy up using the fact that $i^2=-1$.]

$$\Rightarrow\frac{5i-5}{-2}=5a-5ai$$ [Multiply both sides by -2.]

$$\Rightarrow 5i-5=-2(5a-5ai)$$

$$-5+5i=-10a+10ai$$

Equate the real parts: $-5=-10a\Rightarrow a=\frac{-5}{-10}=\frac{1}{2}$