Complex Numbers (Q 4, Paper 1)

Lesson No. 5: Modulus

2006

4 (a) Let $u = 3 - 6i$ where $i^2 = -1$.
Calculate $|u + 2i|$.

Solution

4 (a)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ **2**

$|u + 2i| = |3 - 6i + 2i| = |3 - 4i|$

$= \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

2005

4 (b) Let $w = 1 + 3i$.

(i) Express $\frac{2}{w}$ in the form $x + yi$, where $x, y \in \mathbf{R}$.

(ii) Investigate whether $|iw + w| = |iw| + |w|$.

Solution

4 (b) (i)

$w = 1 = 3i$

Division: Multiply above and below by the conjugate of the bottom.

$\frac{2}{w} = \frac{2}{1 + 3i}$ [Multiply above and below by the conjugate.]

$= \frac{2}{(1 + 3i)} \times \frac{(1 - 3i)}{(1 - 3i)}$ [Multiply out the brackets.]

$= \frac{2 - 6i}{1 - 3i + 3i - 9i^2}$ [Tidy up the bottom using the fact that $i^2 = -1$.]

$= \frac{2 - 6i}{1 + 9}$ [Divide the bottom number into each term on top.]

$= \frac{2 - 6i}{10} = \frac{1}{5} - \frac{3}{5}i$

Cont...

4 (b) (ii)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ 2

LHS

$|iw + w| = |i(1+3i) + (1+3i)|$

$= |i + 3i^2 + 1 + 3i|$

$= |i - 3 + 1 + 3i|$

$= |-2 + 4i|$

$= \sqrt{(-2)^2 + (4)^2}$

$= \sqrt{4 + 16} = \sqrt{20}$

$= 2\sqrt{5}$

RHS

$|iw| + |w| = |i(1+3i)| + |1+3i|$

$= |i + 3i^2| + |1 + 3i|$

$= |-3 + i| + |1 + 3i|$

$= \sqrt{(-3)^2 + (1)^2} + \sqrt{(1)^2 + (3)^3}$

$= \sqrt{10} + \sqrt{10}$

$= 2\sqrt{10}$

Therefore, this statement is false.

2003

4 (b) Let $z_1 = 2 + 3i$ and $z_2 = 5 - i$.

(i) Plot z_1 and z_2 and $z_1 + z_2$ on an Argand diagram.

(ii) Investigate whether $|z_1 + z_2| > |z_1 - z_2|$.

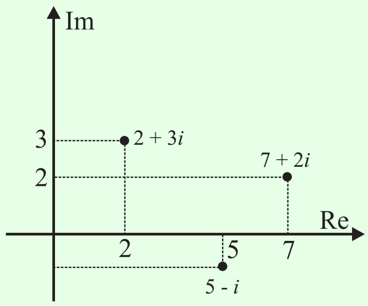
Solution

4 (b) (i)

$z_1 = 2 + 3i$

$z_2 = 5 - i$

$z_1 + z_2 = 2 + 3i + 5 - i = 7 + 2i$



4 (b) (ii)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ 2

LHS

$|z_1 + z_2| = |7 + 2i|$

$= \sqrt{7^2 + 2^2} = \sqrt{49 + 4}$

$= \sqrt{53}$

RHS

$|z_1 - z_2| = |2 + 3i - 5 + i|$

$= |-3 + 4i|$

$= \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16}$

$= \sqrt{25}$

As $\sqrt{53} > \sqrt{25}$, the statement is true.

2001

4 (c) Let $z_1 = 3 + 4i$ and $z_2 = 12 - 5i$.

$\overline{z}_1$ and $\overline{z}_2$ are the complex conjugates of z_1 and z_2, respectively.

(i) Show that $z_1\overline{z}_2 + \overline{z}_1 z_2$ is a real number.

(ii) Investigate if $|z_1| + |z_2| = |z_1 + z_2|$.

SOLUTION

4 (c) (i)

Working out the conjugate: $z = a + bi \Rightarrow \overline{z} = a - bi$ **1**

$z_1\overline{z}_2 + \overline{z}_1 z_2$

$= (3+4i)(12+5i) + (3-4i)(12-5i)$ [Multiply out the brackets.]

$= 36 + 15i + 48i + 20i^2 + 36 - 15i - 48i + 20i^2$ [Tidy up using the fact that $i^2 = -1$.]

$= 36 - 20 + 36 - 20$

$= 32$

4 (c) (ii)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ **2**

LHS

$|z_1| + |z_2|$

$= |3+4i| + |12-5i|$

$= \sqrt{3^2 + 4^2} + \sqrt{12^2 + (-5)^2}$

$= \sqrt{9+16} + \sqrt{144+25}$

$= \sqrt{25} + \sqrt{169}$

$= 5 + 13$

$= 18$

RHS

$|z_1 + z_2|$

$= |3 + 4i + 12 - 5i|$

$= |15 - i|$

$= \sqrt{15^2 + (-1)^2}$

$= \sqrt{225 + 1}$

$= \sqrt{226}$

$= 15.03$

Therefore, the statement is not true.

2000

4 (b) Let $w = 3 - i$.

(i) Plot w and $w + 6i$ on an Argand diagram.

(ii) Calculate $|w + 6i|$.

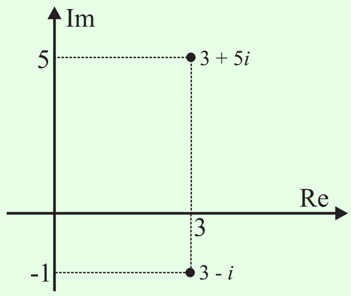
(iii) Express $\frac{1}{w+6i}$ in the form $u + vi$ where $u, v \in \mathbf{R}$.

SOLUTION

4 (b) (i)

$w = 3 - i$

$w + 6i = 3 - i + 6i = 3 + 5i$



4 (b) (ii)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ 2

$|w + 6i| = |3 + 5i| = \sqrt{3^2 + 5^2}$

$= \sqrt{9 + 25} = \sqrt{41}$

4 (b) (iii)

Working out the conjugate: $z = a + bi \Rightarrow \bar{z} = a - bi$ 1

DIVISION: Multiply above and below by the conjugate of the bottom.

$\frac{1}{w+6i} = \frac{1}{3+5i}$ [Multiply above and below by the conjugate of the bottom.]

$= \frac{1}{(3+5i)} \times \frac{(3-5i)}{(3-5i)}$ [Multiply out the brackets.]

$= \frac{3-5i}{9-15i+15i-25i^2}$ [Tidy up using the fact that $i^2 = -1$.]

$= \frac{3-5i}{9+25} = \frac{3-5i}{34}$

$= \frac{3}{34} - \frac{5}{34}i$

1999

4 (b) Let $u = 3 - 6i$.

(i) Calculate $|u|$.

(ii) Show that $iu + \frac{u}{i} = 0$.

(iii) Express $\frac{u}{u+3i}$ in the form $p + qi, \quad p, q \in \mathbf{R}$.

SOLUTION

4 (b) (i)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ 2

$|u| = |3 - 6i| = \sqrt{3^2 + (-6)^2}$

$= \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}$

4 (b) (ii)

$iu + \frac{u}{i} = i(3 - 6i) + \frac{3 - 6i}{i}$

$= 3i - 6i^2 + \frac{(3 - 6i)}{i} \times \frac{i}{i}$ [If i is on the bottom of a fraction, multiply above and below by i.]

$= 3i + 6 + \frac{3i - 6i^2}{i^2}$ [Tidy up using the fact that $i^2 = -1$.]

$= 3i + 6 + \frac{3i + 6}{-1}$

$= 3i + 6 - 3i - 6$

$= 0$

4 (b) (iii)

Working out the conjugate: $z = a + bi \Rightarrow \bar{z} = a - bi$ 1

DIVISION: Multiply above and below by the conjugate of the bottom.

$\frac{u}{u + 3i} = \frac{3 - 6i}{3 - 6i + 3i}$

$= \frac{3 - 6i}{3 - 3i}$ [Divide each term above and below by 3.]

$= \frac{1 - 2i}{1 - i}$ [Multiply above and below by the conjugate of the bottom.]

$= \frac{(1 - 2i)}{(1 - i)} \times \frac{(1 + i)}{(1 + i)}$ [Multiply out the brackets.]

$= \frac{1 + i - 2i - 2i^2}{1 + i - i - i^2}$ [Tidy up using the fact that $i^2 = -1$.]

$= \frac{1 - i + 2}{1 + 1} = \frac{3 - i}{2}$ [Divide the 2 on the bottom into each term above.]

$= \frac{3}{2} - \frac{1}{2}i$

1998

4 (b) (ii) Investigate if

$$|2+14i| = |10(1-i)|.$$

SOLUTION

Finding the modulus:

$$z = a+bi \Rightarrow |z| = \sqrt{a^2+b^2}$$ **2**

LHS

$|2+14i| = \sqrt{2^2+14^2}$

$= \sqrt{4+196} = \sqrt{200}$

$= \sqrt{100\times 2} = 10\sqrt{2}$

RHS

$|10(1-i)| = |10-10i|$

$= \sqrt{10^2+(-10)^2}$

$= \sqrt{100+100} = \sqrt{200}$

$= \sqrt{100\times 2} = 10\sqrt{2}$

Therefore, the statement is true.

1997

4 (b) (i) For what values of a is

$$|a+8i| = 10 \text{ where } a\in \mathbf{R}?$$

SOLUTION

For all equations you can equate (set equal) the real parts and the imaginary parts.

Finding the modulus:

$$z = a+bi \Rightarrow |z| = \sqrt{a^2+b^2}$$ **2**

$|a+8i| = 10$

$\Rightarrow \sqrt{a^2+8^2} = 10$ [Square both sides.]

$\Rightarrow a^2+64 = 100$

$\Rightarrow a^2 = 100-64 = 36$

$\Rightarrow a = \pm\sqrt{36} = \pm 6$