Complex Numbers (Q 4, Paper 1)

1998

4 (a) Let $w = 2i$, where $i^2 = -1$. Plot

(i) w^2,

(ii) w^3

on an Argand diagram.

(b) (i) Verify that $4-3i$ is a root of

$$z^2 - 8z + 25 = 0$$

and write down the other root.

(ii) Investigate if

$$|2+14i| = |10(1-i)|.$$

(c) Let $u = 2 - i$.

(i) Express $u + \frac{1}{u}$ in the form $a + bi$, $a, b \in \mathbf{R}$.

(ii) Hence, solve

$$k(u + \tfrac{1}{u}) + ti = 18$$

for real k and real t.

Solution

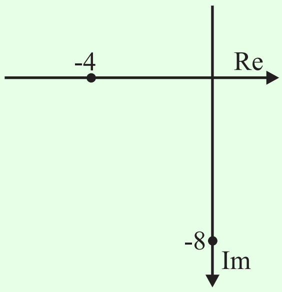
4 (a) (i)

$w^2 = (2i)^2 = 4i^2 = -4 = -4 + 0i$

4 (a) (ii)

$w^3 = (2i)^3 = 8i^3 = -8i = 0 - 8i$

> Powers of i
> $i = \sqrt{-1} = i$
> $i^2 = -1$
> $i^3 = -i$
> $i^4 = 1$



4 (b) (i)

> If $a + bi$ is a root of a quadratic equation with all real coefficients, then its conjugate, $a - bi$, is also a root.

To show that that $4-3i$ is a root of $z^2 - 8z + 25 = 0$ substitute it in for z and show that you get zero.

$(4-3i)^2 - 8(4-3i) + 25$

$= (4-3i)(4-3i) - 8(4-3i) + 25$

$= 16 - 12i - 12i + 9i^2 - 32 + 24i + 25$

$= 16 - 12i - 12i - 9 - 32 + 24i + 25$

$= 0$

Therefore, $4-3i$ is a root. The other root is $4 + 3i$.

4 (b) (ii)

Finding the modulus:

$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ **2**

LHS

$|2+14i| = \sqrt{2^2 + 14^2}$

$= \sqrt{4+196} = \sqrt{200}$

$= \sqrt{100 \times 2} = 10\sqrt{2}$

RHS

$|10(1-i)| = |10-10i|$

$= \sqrt{10^2 + (-10)^2}$

$= \sqrt{100+100} = \sqrt{200}$

$= \sqrt{100 \times 2} = 10\sqrt{2}$

Therefore, the statement is true.

4 (c) (i)

Working out the conjugate: $z = a + bi \Rightarrow \bar{z} = a - bi$ **1**

Division: Multiply above and below by the conjugate of the bottom.

$u + \frac{1}{u} = 2 - i + \frac{1}{2-i}$

$= 2 - i + \frac{1}{(2-i)} \times \frac{(2+i)}{(2+i)}$ [Multiply above and below by the conjugate of the bottom.]

$= 2 - i + \frac{2+i}{4 + 2i - 2i - i^2}$ [Multiply out the brackets.]

$= 2 - i + \frac{2+i}{4+1} = 2 - i + \frac{2+i}{5}$ [Tidy up using the fact that $i^2 = -1$.]

$= 2 - i + \frac{2}{5} + \frac{1}{5}i$ [Divide the 5 on the bottom into each term above.]

$= \frac{12}{5} - \frac{4}{5}i$ [Add the real parts and the imaginary parts.]

4 (c) (ii)

For all equations you can equate (set equal) the real parts and the imaginary parts.

$k\left(u + \frac{1}{u}\right) + ti = 18$ [Write 18 as a complex number.]

$\Rightarrow k(\frac{12}{5} - \frac{4}{5}i) + ti = 18 + 0i$

$\Rightarrow \frac{12}{5}k - \frac{4}{5}ki + ti = 18 + 0i$

$\Rightarrow \frac{12}{5}k + (-\frac{4}{5}k + t)i = 18 + 0i$ [Gather up the real parts and the imaginary parts.]

Equating the real parts: $\frac{12}{5}k = 18 \Rightarrow k = 18 \times \frac{5}{12} = \frac{15}{2}$

Equating the imaginary parts: $-\frac{4}{5}k + t = 0 \Rightarrow -\frac{4}{5}(\frac{15}{2}) + t = 0 \Rightarrow -6 + t = 0 \Rightarrow t = 6$