COMPLEX NUMBERS (Q 4, PAPER 1)

1996

4 (a) Let $z = 1 - 4i$, where $i^2 = -1$.
Plot z and $2 + z$ on an Argand diagram.

(b) Let $w = (1-3i)(2+i)$.

Express w in the form $p + qi$, $p, q \in \mathbf{R}$.

Verify that

$$|w + \bar{w}| = |w - \bar{w}|,$$

where $\bar{w}$ is the complex conjugate of w.

For what value of a is

$$\frac{\bar{w}}{2i} = aw,$$

where $a \in \mathbf{R}$?

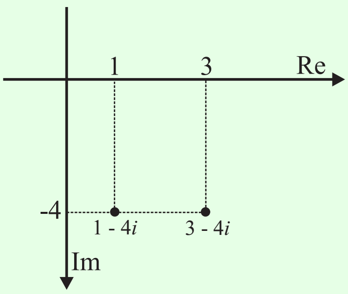
(c) Let $z = 2 - i$ be one root of the equation $z^2 + pz + q = 0$, $p, q \in \mathbf{R}$.
Find the value of p and the value of q.

SOLUTION

4 (a)

$z = 1 - 4i$

$2 + z = 2 + 1 - 4i = 3 - 4i$



4 (b)

$w = (1-3i)(2+i)$ [Multiply out the brackets.]

$= 2 + i - 6i - 3i^2$ [Tidy up using the fact that $i^2 = -1$.]

$= 2 + i - 6i + 3$ [Add the real parts and the imaginary parts.]

$= 5 - 5i$

Working out the conjugate: $z = a + bi \Rightarrow \bar{z} = a - bi$ **1**

Finding the modulus: $z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$ **2**

$w = 5 - 5i \Rightarrow \bar{w} = 5 + 5i$

LHS	*LHS*
$\lvert w + \bar{w} \rvert$	$\lvert w - \bar{w} \rvert$
$= \lvert 5 - 5i + 5 + 5i \rvert$	$= \lvert 5 - 5i - 5 - 5i \rvert$
$= \lvert 10 + 0i \rvert$	$= \lvert 0 - 10i \rvert$
$= \sqrt{10^2 + 0^2}$	$= \sqrt{0^2 + (-10)^2} = \sqrt{0 + 100}$
$= \sqrt{100} = 10$	$= \sqrt{100} = 10$

For all equations you can equate (set equal) the real parts and the imaginary parts.

$$\frac{\bar{w}}{2i} = aw$$

$$\Rightarrow \frac{5+5i}{2i} = a(5-5i)$$

$$\Rightarrow \frac{(5+5i)}{2i} \times \frac{i}{i} = 5a - 5ai$$

$$\Rightarrow \frac{5i+5i^2}{2i^2} = 5a - 5ai$$ [Tidy up using the fact that $i^2 = -1$.]

$$\Rightarrow \frac{5i-5}{-2} = 5a - 5ai$$ [Multiply both sides by -2.]

$$\Rightarrow 5i - 5 = -2(5a - 5ai)$$

$$-5 + 5i = -10a + 10ai$$

Equate the real parts: $-5 = -10a \Rightarrow a = \frac{-5}{-10} = \frac{1}{2}$

4 (c)

If $a + bi$ is a root of a quadratic equation with all real coefficients, then its conjugate, $a - bi$, is also a root.

If $2 - i$ is a root of $z^2 + pz + q = 0$, then $2 + i$ is also a root.
Substitute one of the roots into the quadratic.

$$(2-i)^2 + p(2-i) + q = 0$$

$$\Rightarrow (2-i)(2-i) + p(2-i) + q = 0$$

$$\Rightarrow 4 - 2i - 2i + i^2 + 2p - pi + q = 0$$

$$\Rightarrow 4 - 4i - 1 + 2p - pi + q = 0$$

$$\Rightarrow (3 + 2p + q) + (-p - 4)i = 0 + 0i$$ [Gather up the real parts and the imaginary parts.]

Equate the imaginary parts: $-p - 4 = 0 \Rightarrow p = -4$

Equate the real parts: $3 + 2p + q = 0 \Rightarrow 3 + 2(-4) + q = 0 \Rightarrow 3 - 8 + q = 0$

$$\Rightarrow -5 + q = 0 \Rightarrow q = 5$$