## The Circle (Q 3, Paper 2)

## Lesson No. 6: Right-Angled triangles inside circles

## 2006

3 (b) The vertices of a right-angled triangle are $p(1,1), q(5,1)$ and $r(1,4)$.
The circle $K$ passes through the points $p, q$ and $r$.
(i) On a coordinate diagram, draw the triangle pqr.

Mark the point $c$, the centre of $K$, and draw $K$.
(ii) Find the equation of $K$.
(iii) Find the equation of the image of $K$ under the translation $(5,1) \rightarrow(1,4)$.

## Solution

3 (b) (i)


3 (b) (ii)
The angle in a semi-circle at $c$ is a right-angle $\left(90^{\circ}\right)$. To prove this you need to show that $a c$ is perpendicular to $b c$.

Slope of $a c \times$ Slope of $b c=-1 \Rightarrow a c \perp b c$
This means that $[a b]$ is a diameter of a circle.


The centre $c$ is the midpoint of $[r q]$.
The formula for the midpoint, $c$, of the line segment [ab] is:

$$
\text { Midpoint }=\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right)
$$

$2 a\left(x_{1}, y_{1}\right)$

Remember the midpoint formula as: Midpoint $=\left(\frac{\text { Add the } x^{\prime} \text { s }}{2}, \frac{\text { Add the } y^{\prime} \mathrm{s}}{2}\right)$

$$
\begin{array}{rr}
r(1,4) & q(5,1) \\
\downarrow \downarrow & \downarrow \downarrow \\
x_{1} y_{1} & x_{2} y_{2}
\end{array}
$$

Midpoint of $[r q]=\left(\frac{1+5}{2}, \frac{4+1}{2}\right)=\left(\frac{6}{2}, \frac{5}{2}\right)=c\left(3, \frac{5}{2}\right)$

The radius $r$ is the distance $|c q|$.

$$
d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \ldots \ldots
$$

The distance between $a$ and $b$ is written as $|a b|$.
REMEMBER THE DISTANCE FORMULA AS:


$$
d=\sqrt{\left(\text { Difference in } x^{\prime} \mathrm{s}\right)^{2}+\left(\text { Difference in } y^{\prime} \mathrm{s}\right)^{2}}
$$

$$
\left.\begin{array}{cc|}
c c \mid \\
c\left(3, \frac{5}{2}\right) & q(5,1) \\
\downarrow \downarrow & \downarrow \downarrow \\
x_{1} y_{1} & x_{2} y_{2}
\end{array}\right] \Rightarrow \begin{aligned}
& |c q|=r=\sqrt{(5-3)^{2}+\left(1-\frac{5}{2}\right)^{2}} \\
&
\end{aligned}
$$

Circle $C$ with centre $(h, k)$, radius $r$.

$$
\begin{equation*}
(x-h)^{2}+(y-k)^{2}=r^{2} \tag{2}
\end{equation*}
$$

To get the centre: Change the sign of the number inside each bracket. To get the radius: Take the square root of the number on the right.

Equation of $K$ : centre $c\left(3, \frac{5}{2}\right), r=\frac{5}{2}$

$$
\begin{aligned}
& K:(x-3)^{2}+\left(y-\frac{5}{2}\right)^{2}=\left(\frac{5}{2}\right)^{2} \\
& \therefore(x-3)^{2}+\left(y-\frac{5}{2}\right)^{2}=\frac{25}{4}
\end{aligned}
$$

## 3 (b) (iii)

The circle $K$ remains unchnged under a translation. Its location changes. Find its new centre as shown on the right.
Image of $K$ : centre $\left(-1, \frac{11}{2}\right), r=\frac{5}{2}$
$(x+1)^{2}+\left(y-\frac{11}{2}\right)^{2}=\left(\frac{5}{2}\right)^{2}$
$\therefore(x+1)^{2}+\left(y-\frac{11}{2}\right)^{2}=\frac{25}{4}$


## 2004

3 (b) A circle has equation $x^{2}+y^{2}=13$.
The points $a(2,-3), b(-2,3)$ and $c(3,2)$ are on the circle.
(i) Verify that $[a b]$ is a diameter of the circle.
(ii) Verify that $\angle a c b$ is a right angle.

## Solution

3 (b) (i)
To show a line segment is a diameter of a circle:
The midpoint of a diameter is the centre of a circle.
The centre of the circle $x^{2}+y^{2}=13$ is $(0,0)$.
The formula for the midpoint, $c$, of the line segment $[a b]$ is:

$$
\text { Midpoint } \left.=\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right) \right\rvert\, \ldots \ldots .2 a\left(x_{1}, y_{1}\right)
$$



Remember the midpoint formula as: Midpoint $=\left(\frac{\text { Add the } x^{\prime} \text { s }}{2}, \frac{\text { Add the } y^{\prime} \text { s }}{2}\right)$

| $a(2$, | $-3)$ | $b(-2$, | $3)$ |
| :---: | :---: | :---: | :---: |
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $x_{1}$ | $y_{1}$ | $x_{2}$ | $y_{2}$ |

Midpoint of $[a b]=\left(\frac{2-2}{2}, \frac{-3+3}{2}\right)=\left(\frac{0}{2}, \frac{0}{2}\right)=(0,0)$
The midpoint of $[a b]$ is the centre of the circle. Therefore, [ab] is a diameter of the circle.

3 (b) (ii)
The angle in a semi-circle at $c$ is a right-angle $\left(90^{\circ}\right)$. To prove this you need to show that $a c$ is perpendicular to $b c$.

$$
\text { Slope of } a c \times \text { Slope of } b c=-1 \Rightarrow a c \perp b c
$$

This means that $[a b]$ is a diameter of a circle.


You can do this 3 ways:

1. Find the slope of $a c$ and $b c$ and show they are perpendicular.
2. Find the lengths of the 3 sides and apply Pythagoras' theorem.
3. Show $c$ is on the circle. Any angle standing on the diameter is a right angle. This is my favourite and by far the quickest.
$c(3,2) \in x^{2}+y^{2}=13$ ?
$(3)^{2}+(2)^{2}=9+4$
$=13 \Rightarrow x^{2}+y^{2}=13$
$\therefore \angle a c b$ is a right-angle.

## 2002

3 (c) $a(-5,1), b(3,7)$ and $c(9,-1)$ are three points.
(i) Show that the triangle $a b c$ is right-angled.
(ii) Hence, find the centre of the circle that passes through $a, b$ and $c$ and write down the equation of the circle.

## Solution

## 3 (c) (i)

Find the slope of all 3 sides and show that two of the sides are perpendicular.

$$
m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \ldots \ldots .3 \quad \begin{aligned}
& \text { REMEMBER IT AS: } \\
& \text { Slope } m=\frac{\text { Difference in } y^{\prime} \mathrm{s}}{\text { Difference in } x^{\prime} \mathrm{s}}
\end{aligned}
$$

$$
\begin{array}{ccc}
a(-5,1) & b(3,7) \\
\downarrow & \downarrow & \downarrow \\
x_{1} & y_{1} & x_{2} \\
\hline
\end{array} y_{2} .
$$

Slope of $a b: m_{1}=\frac{7-1}{3-(-5)}=\frac{6}{3+5}=\frac{6}{8}=\frac{3}{4}$

$$
\begin{array}{ccc}
a(-5,1) & c(9, & -1) \\
\downarrow & \downarrow & \downarrow \\
x_{1} & y_{1} & x_{2} \\
y_{2}
\end{array}
$$

Slope of $a c: m_{2}=\frac{-1-1}{9-(-5)}=\frac{-2}{9+5}=\frac{-2}{14}=-\frac{1}{7}$

$$
\begin{array}{cccc}
b(3,7) & c(9, & -1) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
x_{1} & y_{1} & x_{2} & y_{2}
\end{array}
$$

Slope of $b c$ : $m_{3}=\frac{-1-7}{9-3}=\frac{-8}{6}=-\frac{4}{3}$

> Two lines are perpendicular if the product of their slopes is -1 .

$$
m_{1} \times m_{3}=\left(\frac{3}{4}\right)\left(-\frac{4}{3}\right)=-1 \Rightarrow a b \perp b c
$$

Therefore, the triangle $a b c$ is right-angled with the right angle at $b$.
3 (c) (ii)

The angle in a semi-circle at $b$ is a right-angle $\left(90^{\circ}\right)$. This means that $[a c]$ is a diameter of a circle.

The centre $o$ is the midpoint of $a c$.


The formula for the midpoint, $c$, of the line segment [ab] is:

$$
\text { Midpoint }=\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right)
$$

2


Remember the midpoint formula as: Midpoint $=\left(\frac{\text { Add the } x^{\prime} \text { s }}{2}, \frac{\text { Add the } y \text { 's }}{2}\right)$

Cont....

$$
\begin{array}{rcc|}
a(-5,1) & c(9,-1) \\
\downarrow & \downarrow & \downarrow \\
x_{1} & y_{1} & x_{2}
\end{array} y_{2} . \quad \text { Midpoint } o=\left(\frac{-5+9}{2}, \frac{1-1}{2}\right)=\left(\frac{4}{2}, \frac{0}{2}\right)=(2,0)
$$

The radius is the distance from the centre $o$ to any vertex on the triangle, say $b(3,7)$.

$$
d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \ldots \ldots .
$$

The distance between $a$ and $b$ is written as $|a b|$.
Remember the distance formula as:


$$
d=\sqrt{\left(\text { Difference in } x^{\prime} \mathrm{s}\right)^{2}+\left(\text { Difference in } y^{\prime} \mathrm{s}\right)^{2}}
$$

$$
\begin{array}{|cc|}
\begin{array}{ccc}
o(2,0) & b(3,7) \\
\downarrow \downarrow & \downarrow \downarrow \\
x_{1} y_{1} & x_{2} y_{2}
\end{array} & \begin{aligned}
& \Rightarrow|o b|=\sqrt{(1)^{2}+(7)^{2}} \\
& \Rightarrow|o b|=\sqrt{1+49} \\
& \therefore r=\sqrt{50}
\end{aligned}
\end{array}
$$

Equation of the circle: centre $(h, k)=(2,0), r=\sqrt{50}$

Circle $C$ with centre $(h, k)$, radius $r$.

$$
\begin{equation*}
(x-h)^{2}+(y-k)^{2}=r^{2} \tag{2}
\end{equation*}
$$

To get the centre: Change the sign of the number inside each bracket.
To get the radius: Take the square root of the number on the right.
$(x-h)^{2}+(y-k)^{2}=r^{2}$
$\Rightarrow(x-2)^{2}+(y-0)^{2}=(\sqrt{50})^{2}$
$\therefore(x-2)^{2}+y^{2}=50$

