THE CIRCLE (Q 3, PAPER 2)

2006

3 (a) The circle C has equation $x^2 + y^2 = 25$.

The line L is a tangent to C at the point $(-3, 4)$.

(i) Verify that the point $(-3, 4)$ is on C.

(ii) Find the slope of L.

(iii) Find the equation of L.

(iv) The line T is another tangent to C and is parallel to L.
Find the coordinates of the point at which T touches C.

(b) The vertices of a right-angled triangle are $p(1, 1)$, $q(5, 1)$ and $r(1, 4)$.
The circle K passes through the points p, q and r.

(i) On a coordinate diagram, draw the triangle pqr.
Mark the point c, the centre of K, and draw K.

(ii) Find the equation of K.

(iii) Find the equation of the image of K under the translation $(5, 1) \rightarrow (1, 4)$.

SOLUTION

3 (a) (i)

IS A POINT ON A CIRCLE, INSIDE A CIRCLE OR OUTSIDE A CIRCLE?
Substitute the point into the circle.
On the circle: Both sides are equal.
Inside the circle: The left hand side is less than the right hand side.
Outside the circle: The left hand side is greater than the right hand side.

$(-3, 4) \in x^2 + y^2 = 25?$

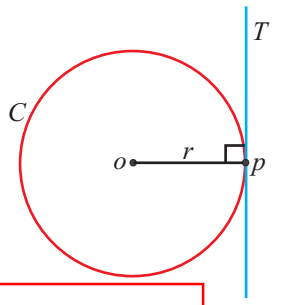
$(-3)^2 + (4)^2 = 9 + 16$

$= 25 \Rightarrow (-3, 4) \in x^2 + y^2 = 25$

3 (a) (ii)

SOME POINTS ABOUT TANGENTS:

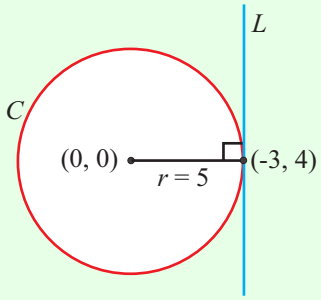
- There is only one point of contact, p, between the tangent, T, and the circle, C.
- The slope of the tangent is perpendicular to the line joining the centre to the point of contact, op.



FINDING THE EQUATION OF A TANGENT, T, TO A CIRCLE:

STEPS
1. Find the slope of the line from the centre to the point of contact.
2. Find the slope of the tangent (it is perpendicular to the radius).
3. Find the equation of T.

First, find the slope between the centre (0, 0) and the point of contact $(-3,\ 4)$.



$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{.....} \quad 3$$

REMEMBER IT AS:

Slope $m = \dfrac{\text{Difference in } y\text{'s}}{\text{Difference in } x\text{'s}}$

$$\begin{array}{cc} (0,\ 0) & (-3,\ 4) \\ \downarrow\downarrow & \downarrow\ \downarrow \\ x_1\ y_1 & x_2\ y_2 \end{array}$$

Slope: $m = \dfrac{4-0}{-3-0} = \dfrac{4}{-3} = -\dfrac{4}{3}$

FINDING THE PERPENDICULAR SLOPE: Invert the slope and change its sign.

The tangent L is perpendicular to this slope.

Slope of L: $m = \frac{3}{4}$

3 (a) (iii)

Equation of L: Point $(x_1,\ y_1) = (-3,\ 4)$, slope $m = \frac{3}{4}$.

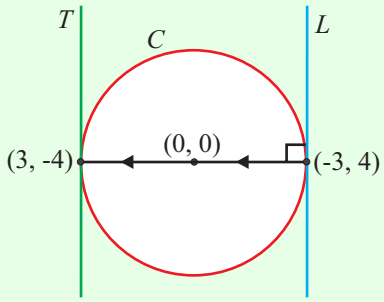
The equation of a line is a formula satisfied by every point (x, y) on the line.

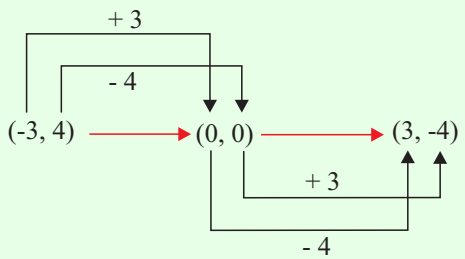
Equation of a line: $y - y_1 = m(x - x_1)$ 4

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 4 = \tfrac{3}{4}(x - (-3))$$
$$\Rightarrow y - 4 = \tfrac{3}{4}(x + 3)$$
$$\Rightarrow 4(y - 4) = 3(x + 3)$$
$$\Rightarrow 4y - 16 = 3x + 9$$
$$\therefore 3x - 4y + 25 = 0$$

3 (a) (iv)

FINDING A PARALLEL TANGENT TO A CIRCLE:
A tangent to a circle, T_1, has a parallel tangent, T_2, on the other side of the circle. The centre, o, is the midpoint of their points of contact, p and q. The slopes of the two tangents are the same.

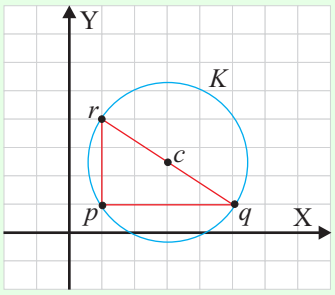




$(-3,\ 4) \rightarrow (0,\ 0) \rightarrow (3,\ -4)$

ANS: $(3,\ -4)$

3 (b) (i)

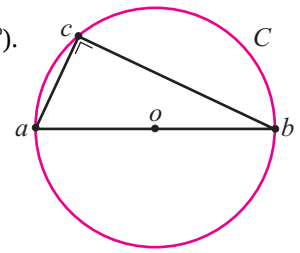


3 (b) (ii)

The angle in a semi-circle at c is a right-angle (90°). To prove this you need to show that ac is perpendicular to bc.

$$\boxed{\text{Slope of } ac \times \text{ Slope of } bc = -1 \Rightarrow ac \perp bc}$$

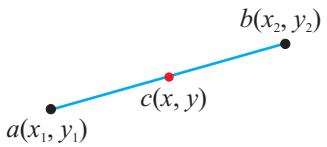
This means that $[ab]$ is a diameter of a circle.



The centre c is the midpoint of $[rq]$.

The formula for the midpoint, c, of the line segment $[ab]$ is:

$$\boxed{\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)} \quad \ldots\ldots \; 2$$



REMEMBER THE MIDPOINT FORMULA AS: Midpoint $= \left(\dfrac{\text{Add the } x\text{'s}}{2}, \dfrac{\text{Add the } y\text{'s}}{2}\right)$

$$\boxed{\begin{array}{cc} r(1,\ 4) & q(5,\ 1) \\ \downarrow\downarrow & \downarrow\downarrow \\ x_1\ y_1 & x_2\ y_2 \end{array}}$$

$$\text{Midpoint of } [rq] = \left(\frac{1+5}{2}, \frac{4+1}{2}\right) = \left(\frac{6}{2}, \frac{5}{2}\right) = c(3,\ \tfrac{5}{2})$$

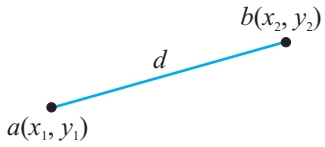
The radius r is the distance $|cq|$.

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \quad \ldots\ldots \; 1$$

The distance between a and b is written as $|ab|$.

REMEMBER THE DISTANCE FORMULA AS:

$$d = \sqrt{(\text{Difference in } x\text{'s})^2 + (\text{Difference in } y\text{'s})^2}$$



$$\boxed{\begin{array}{cc} c(3,\ \tfrac{5}{2}) & q(5,\ 1) \\ \downarrow\downarrow & \downarrow\downarrow \\ x_1\ y_1 & x_2\ y_2 \end{array}}$$

$$|cq| = r = \sqrt{(5-3)^2 + (1-\tfrac{5}{2})^2}$$

$$\Rightarrow r = \sqrt{(2)^2 + (-\tfrac{3}{2})^2} = \sqrt{4 + \tfrac{9}{4}}$$

$$\therefore r = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2}$$

Circle C with centre (h, k), radius r.

$$\boxed{(x-h)^2+(y-k)^2=r^2} \quad \ldots\ldots \; 2$$

To get the centre: Change the sign of the number inside each bracket.
To get the radius: Take the square root of the number on the right.

Equation of K: centre $c(3, \frac{5}{2})$, $r=\frac{5}{2}$

$K:(x-3)^2+(y-\frac{5}{2})^2=(\frac{5}{2})^2$

$\therefore (x-3)^2+(y-\frac{5}{2})^2=\frac{25}{4}$

3 (b) (iii)

The circle K remains unchnged under a translation. Its location changes. Find its new centre as shown on the right.

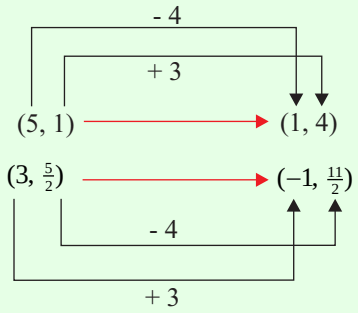


Image of K: centre $(-1, \frac{11}{2})$, $r=\frac{5}{2}$

$(x+1)^2+(y-\frac{11}{2})^2=(\frac{5}{2})^2$

$\therefore (x+1)^2+(y-\frac{11}{2})^2=\frac{25}{4}$