ALGEBRA (Q 2 & 3, PAPER 1)

LESSON NO. 9: FUNCTIONS

2006

2 (b) Let $f(x)=2x^3+ax^2+bx+14$.

(i) Express $f(2)$ in terms of a and b.

(ii) If $f(2)=0$ and $f(-1)=0$, find the value of a and the value of b.

SOLUTION

2 (b) (i)

$f(x)=2x^3+ax^2+bx+14$

$\Rightarrow f(2)=2(2)^3+a(2)^2+b(2)+14=16+4a+2b+14$

$\therefore f(2)=4a+2b+30$

2 (b) (ii)

$f(2)=0 \Rightarrow 4a+2b+30=0 \Rightarrow 2a+b=-15....(\mathbf{1})$

$f(-1)=0 \Rightarrow 2(-1)^3+a(-1)^2+b(-1)+14=0$

$\Rightarrow -2+a-b+14=0 \Rightarrow a-b=-12...(\mathbf{2})$

Solve equations (**1**) and (**2**) simultaneously.

$$\begin{aligned} 2a+b&=-15...(\mathbf{1}) \\ a-b&=-12....(\mathbf{2}) \\ \hline 3a\quad&=-27 \Rightarrow a=-9 \end{aligned}$$

Substitute this value of a into equation (**2**): $(-9)-b=-12 \Rightarrow b=3$

2004

3 (c) p is a positive number and f is the function $f(x)=(2x+p)(x-p),\ x\in \mathbf{R}$.

(i) Given that $f(2) = 0$, find the value of p.

(ii) Hence, find the range of values of x for which $f(x) < 0$.

SOLUTION

3 (c) (i)

$f(x)=(2x+p)(x-p) \Rightarrow f(2)=(4+p)(2-p)=0$

Set each bracket equal to zero and solve for p.

$\therefore p=-4,\ 2$

As $p > 0$, ignore the negative solution.

$\therefore p=2$

3 (c) (ii)

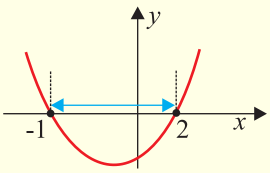
$\therefore f(x)=(2x+2)(x-2)=2x^2-2x-4<0$

Solving quadratic inequalities:

STEPS

1. Find the roots of the quadratic equation: $ax^2+bx+c=0$.
   These are the places where the curve crosses the x-axis.
2. Sketch the graph. It is either $\cup$ shaped or $\cap$ shaped.
3. Use the graph to solve the inequality.
   $y=f(x)>0$ is above the x-axis.
   $y=f(x)<0$ is below the x-axis.

1. Solve $2x^2-2x-4=0 \Rightarrow x^2-x-2=0 \Rightarrow (x-2)(x+1)=0$
   $\therefore x=-1,\ 2$
2. Sketch the graph. The coefficient of x^2 is positive so the graph is $\cup$ shaped.
3. You can see the parts of the graph that are less than zero, i.e. below the x-axis.



2002

3 (c) Let $f(x) = x^2 + ax + t$ where $a, t \in \mathbf{R}$.

(i) Find the value of a, given that $f(-5) = f(-1)$.

(ii) Given that there is only one value of x for which the $f(x) = 0$, find the value of t.

SOLUTION

3 (c) $f(x) = x^2 + ax + t$

3 (c) (i)

$f(-5) = f(-1) \Rightarrow (-5)^2 + a(-5) + t = (-1)^2 + a(-1) + t$

$\Rightarrow 25 - 5a + t = 1 - a + t$

$\Rightarrow 25 - 1 = -a + 5a$

$\Rightarrow 24 = 4a \Rightarrow a = 6$

3 (c) (ii)

$f(x) = 0 \Rightarrow x^2 + 6x + t = 0$

There is only one value of x for which the $f(x) = 0$ means that the quadratic equations has equal roots.

The quadratic equation $ax^2 + bx + c = 0$ has equal roots if $b^2 = 4ac$.

$a = 1$
$b = 6$
$c = t$

$b^2 = 4ac \Rightarrow 6^2 = 4(1)(t) \Rightarrow 36 = 4t$

$\therefore t = 9$

2001

3 (c) Let $f(x)=x^3+ax^2+bx-6$ where a and b are real numbers.

Given that $x-1$ and $x-2$ are factors of $f(x)$

(i) find the value of a and the value of b

(ii) hence, find the values of x for which $f(x) = 0$.

SOLUTION

3 (c)

The factor theorem states that:

If $(x-k)$ is a factor of $f(x)$ then k is a root of $f(x) = 0$,
i.e. $f(k) = 0$ and vice versa.

3 (c) (i)

If $x-1$ is a factor of $f(x)$, then $f(1) = 0$.

$\therefore\ f(1)=(1)^3+a(1)^2+b(1)-6=0$

$\Rightarrow 1+a+b-6=0 \Rightarrow a+b=5.....(\mathbf{1})$

If $x-2$ is a factor of $f(x)$, then $f(2) = 0$.

$\therefore\ f(2)=(2)^3+a(2)^2+b(2)-6=0$

$\Rightarrow 1+a+b-6=0 \Rightarrow a+b=5.....(\mathbf{1})$

$\Rightarrow 8+4a+2b-6=0 \Rightarrow 4a+2b=-2$

$\Rightarrow 2a+b=-1.....(\mathbf{2})$

Solve equations (**1**) and (**2**) simultaneously.

$$\begin{aligned} a+b&=5........(\mathbf{1})\\ 2a+b&=-1.....(\mathbf{2})(\times -1)\end{aligned} \quad \rightarrow \quad \begin{aligned} a+b&=5\\ -2a-b&=1\\ \hline -a\quad &=6 \Rightarrow a=-6\end{aligned}$$

Substitute this value of a into Eqn. (1) $\Rightarrow -6+b=5 \Rightarrow b=11$

ANSWER: $a=-6,\ b=11$

3 (c) (ii)

$f(x)=0 \Rightarrow x^3-6x^2+11x-6=0$

The 2 linear factors multiply to give a quadratic.

$(x-1)(x-2)=x^2-3x+2$

Divide this quadratic into the cubic to get the other linear factor.

$$\begin{array}{r|l} & x-3 \\ x^2-3x+2 & x^3-6x^2+11x-6 \\ & \mp x^3 \pm 3x^2 \mp 2x \\ \hline & -3x^2+9x-6 \\ & \pm 3x^2 \mp 9x \pm 6 \\ \hline & 0 \end{array}$$

$\therefore\ x^3-6x^2+11x-6=(x-1)(x-2)(x-3)=0$

Set each factor equal to zero and solve for x.

$\therefore\ x=1,\ 2,\ 3$

2000

3 (c) (i) $f(x) = ax^2 + bx - 8$, where a and b are real numbers.

If $f(1) = -9$ and $f(-1) = 3$, find the value of a and the value of b.

(ii) Using your values of a and b from **(i)**, find the two values of x for which

$ax^2 + bx = bx^2 + ax$.

SOLUTION

3 (c) (i)

$f(x) = ax^2 + bx - 8$

$f(1) = -9 \Rightarrow a(1)^2 + b(1) - 8 = -9 \Rightarrow a + b = -1....(\mathbf{1})$

$f(-1) = 3 \Rightarrow a(-1)^2 + b(-1) - 8 = 3 \Rightarrow a - b = 11....(\mathbf{2})$

Solve equations (**1**) and (**2**) simultaneously.

$$\begin{aligned} a + b &= -1...(\mathbf{1}) \\ a - b &= 11....(\mathbf{2}) \\ \hline 2a \quad &= 10 \Rightarrow a = 5 \end{aligned}$$

Substitute the value for a back into Eqn. (**1**) $\Rightarrow 5 + b = -1 \Rightarrow b = -6$

3 (c) (ii)

$ax^2 + bx = bx^2 + ax \Rightarrow 5x^2 - 6x = -6x^2 + 5x$ [Bring all terms to the left.]

$\Rightarrow 11x^2 - 11x = 0$ [Factorise the quadratic.]

$\Rightarrow 11x(x - 1) = 0$ [Set each factor equal to zero and solve for x.]

$\therefore x = 0, 1$

1997

3 (c) Let $f(x) = (2+x)(3-x),\ x \in \mathbf{R}$.
Write down the solutions (roots) of $f(x) = 0$.
Let $g(x) = 3x - k$.
The equation $f(x) + g(x) = 0$ has equal roots. Find the value of k.

Solution

$f(x) = (2+x)(3-x)$.

$f(x) = 0 \Rightarrow (2+x)(3-x) = 0$ [Set each factor equal to zero and solve for x.]

$\therefore x = -2, 3$

$g(x) = 3x - k$.

$f(x) + g(x) = 0 \Rightarrow (2+x)(3-x) + 3x - k = 0$

$\Rightarrow 6 + x - x^2 + 3x - k = 0$

$\Rightarrow -x^2 + 4x + (6-k) = 0$

$\Rightarrow x^2 - 4x - (6-k) = 0$

Equal Roots

The quadratic equation $ax^2 + bx + c = 0$ has equal roots if $b^2 = 4ac$.

$b^2 = 4ac \Rightarrow (-4)^2 = 4(1)(k-6)$

$\Rightarrow 16 = 4k - 24$

$\Rightarrow 40 = 4k \Rightarrow k = 10$

$a = 1$
$b = -4$
$c = k - 6$

1996

3 (c) Let $f(x)=(1-x)(2+x),\ x\in\mathbf{R}$.
Write down the solutions of $f(x) = 0$.
Find the range of values of x for which $f(x) > 0$.
Let $g(x)=f(x)-f(x+1)$.
Express $g(x)$ in the form $ax + b,\ a, b \in \mathbf{R}$.
Find the solution set of $g(x) < 0$.

SOLUTION

$f(x)=(1-x)(2+x)$

$f(x)=0 \Rightarrow (1-x)(2+x)=0$ [Set each factor equal to zero and solve for x.]

$\therefore x=-2,\ 1$

To solve quadratic inequalities, you need to sketch the graph of the quadratic function.

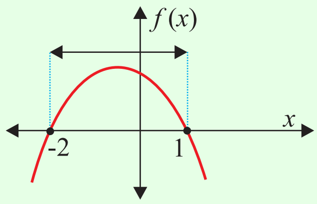
STEPS
1. Find the roots of the quadratic equation: $ax^2+bx+c=0$. These are the places where the curve crosses the x-axis.
2. Sketch the graph. It is either $\cup$ shaped or $\cap$ shaped.
3. Use the graph to solve the inequality.
   $y=f(x)>0$ is above the x-axis.
   $y=f(x)<0$ is below the x-axis.

$f(x)>0 \Rightarrow (1-x)(2+x)>0$ [Multiply out the brackets.]

$\Rightarrow 2+x-x^2>0$

The part of the graph above the x-axis satisfies the inequality.

$\therefore -2<x<1$



$g(x)=f(x)-f(x+1)$

$=2-x-x^2-[2-(x+1)-(x+1)^2]$

$=2-x-x^2-[2-(x+1)-(x^2+2x+1)]$

$=2-x-x^2-[2-x-1-x^2-2x-1]$

$=2-x-x^2-[-3x-x^2]$

$=2-x-x^2+3x+x^2$

$=2x+2$

$g(x)<0 \Rightarrow 2x+2<0$

$\Rightarrow x+1<0$

$\Rightarrow x<-1$