ALGEBRA (Q 2 & 3, PAPER 1)

LESSON NO. 4: CUBIC EQUATIONS

2007

3 (c) Let $f(x) = 2x^3 + 11x^2 + 4x - 5.$

(i) Verify that $f(-1) = 0.$

(ii) Solve the equation

$2x^3 + 11x^2 + 4x - 5 = 0.$

SOLUTION

3 (c) (i)

$f(x) = 2x^3 + 11x^2 + 4x - 5$

$\therefore f(-1) = 2(-1)^3 + 11(-1)^2 + 4(-1) - 5 = 2(-1) + 11(1) - 4 - 5$

$= -2 + 11 - 4 - 5 = 0$

3 (c) (ii)

STEPS

1. Guess at a root (unless a root is given) by substituting in numbers $0, 1, -1, 2, -2, ...$ until you get zero.
2. Using the factor theorem, form a factor from the root.
3. Divide the cubic by the factor to get the quadratic.
4. Solve the quadratic by factorising or using formula **2**.
5. Write down the three roots.

1. $f(-1) = 0 \Rightarrow -1$ is a root.

2. $\therefore (x+1)$ is a factor.

3. Divide the factor into the cubic as shown to the right.

$\therefore 2x^3 + 11x^2 + 4x - 5 = (x+1)(2x^2 + 9x - 5) = 0$

$$\begin{array}{r|l} & 2x^2 + 9x - 5 \\ \hline x+1 & 2x^3 + 11x^2 + 4x - 5 \\ & \mp 2x^3 \mp 2x^2 \\ \hline & \quad 9x^2 + 4x - 5 \\ & \quad \mp 9x^2 \mp 9x \\ \hline & \quad\quad -5x - 5 \\ & \quad\quad \pm 5x \pm 5 \\ \hline & \quad\quad\quad 0 \end{array}$$

4. The resulting quadratic can be factorised.

$2x^2 + 9x - 5 = (2x-1)(x+5)$

5. $2x^3 + 11x^2 + 4x - 5 = (x+1)(2x-1)(x+5) = 0$

Set each factor equal to zero and solve for x.

$\therefore x = -5, -1, \frac{1}{2}$

2005

3 (c) Let $f(x)=2x^3-3x^2-11x+6.$

(i) Verify that $f(3) = 0$.

(ii) Solve the equation

$$2x^3-3x^2-11x+6=0.$$

Solution

3 (c) (i)

$f(x)=2x^3-3x^2-11x+6$

$\therefore f(3)=2(3)^3-3(3)^2-11(3)+6=54-27-33+6=0$

3 (c) (ii)

Steps
1. Guess at a root (unless a root is given) by substituting in numbers $0, 1, -1, 2, -2, ...$ until you get zero.
2. Using the factor theorem, form a factor from the root.
3. Divide the cubic by the factor to get the quadratic.
4. Solve the quadratic by factorising or using formula **2**.
5. Write down the three roots.

1. $f(3)=0 \Rightarrow 3$ is a root.

2. $\therefore (x-3)$ is a factor.

3. Divide the factor into the cubic as shown on the right.

$\therefore 2x^3-3x^2-11x+6=(x-3)(2x^2+3x-2)=0$

$$\begin{array}{r|l} & 2x^2+3x-2 \\ \hline x-3 & 2x^3-3x^2-11x+6 \\ & \mp 2x^3 \pm 6x^2 \\ \hline & \quad 3x^2-11x+6 \\ & \quad \mp 3x^2 \pm 9x \\ \hline & \qquad -2x+6 \\ & \qquad \pm 2x \mp 6 \\ \hline & \qquad\quad 0 \end{array}$$

4. Factorise the quadratic.

$2x^2+3x-2=(2x-1)(x+2)$

5. Write down the three roots.

$\therefore 2x^3-3x^2-11x+6=(x-3)(2x-1)(x+2)=0$

$\therefore x=-2, \frac{1}{2}, 3$

2004

2 (b) (ii) Show that $x-2$ is a factor of x^3-3x^2-x+6.

Solution

The factor theorem states that:

If $(x-k)$ is a factor of $f(x)$ then k is a root of $f(x) = 0$, i.e. $f(k) = 0$ and vice versa.

If $x-2$ is a factor $\Rightarrow 2$ is a root. Substitute 2 in for x.

$f(x)=x^3-3x^2-x+6 \Rightarrow f(2)=(2)^3-3(2)^2-(2)+6$

$=8-12-2+6=0$

Because $f(2)=0 \Rightarrow 2$ is a root. Therefore, $x-2$ is a factor.

2003

3 (b) (i) Show that $x + 2$ is a factor of $x^3 + 3x^2 - 4x - 12$.

(ii) Hence, or otherwise, solve the equation $x^3 + 3x^2 - 4x - 12 = 0$.

SOLUTION

3 (b) (i)

If $(x + 2)$ is a factor $\Rightarrow -2$ is a root. Substitute -2 in for x.

$f(x) = x^3 + 3x^2 - 4x - 12 \Rightarrow f(-2) = (-2)^3 + 3(-2)^2 - 4(-2) - 12$

$= -8 + 12 + 8 - 12 = 0$

Because $f(-2) = 0 \Rightarrow -2$ is a root. Therefore, $(x + 2)$ is a factor.

3 (b) (ii)

STEPS

1. Guess at a root (unless a root is given) by substituting in numbers $0, 1, -1, 2, -2, \ldots$ until you get zero.
2. Using the factor theorem, form a factor from the root.
3. Divide the cubic by the factor to get the quadratic.
4. Solve the quadratic by factorising or using formula **2**.
5. Write down the three roots.

Steps 1 and 2 already done in 3 (b) (i).

3. Divide the cubic by the factor as shown to the right.

$\therefore x^3 + 3x^2 - 4x - 12 = (x + 2)(x^2 + x - 6) = 0$

$$\begin{array}{r|l} & x^2 + x - 6 \\ \hline x + 2 & x^3 + 3x^2 - 4x - 12 \\ & \mp x^3 \mp 2x^2 \\ \hline & \quad\quad x^2 - 4x - 12 \\ & \quad\quad \mp x^2 \mp 2x \\ \hline & \quad\quad\quad -6x - 12 \\ & \quad\quad\quad \pm 6x \pm 12 \\ \hline & \quad\quad\quad\quad 0 \end{array}$$

4. The resulting quadratic can be factorised.

$x^2 + x - 6 = (x + 3)(x - 2)$

5. $\therefore x^3 + 3x^2 - 4x - 12 = (x + 2)(x + 3)(x - 2) = 0$

Set each factor equal to zero and solve for x.

$\therefore x = -3, -2, 2$

2002

2 (b) (i) Show that $x + 2$ is a factor of $2x^3 + 7x^2 + x - 10$.

(ii) Hence, or otherwise, find the three roots of $2x^3 + 7x^2 + x - 10 = 0$.

SOLUTION

2 (b) (i)

The factor theorem states that:

If $(x - k)$ is a factor of $f(x)$ then k is a root of $f(x) = 0$,
i.e. $f(k) = 0$ and vice versa.

If $(x + 1)$ is a factor $\Rightarrow -1$ is a root. Substitute -1 in for x.

$f(x) = x^3 - 2x^2 + 7x + 10 \Rightarrow f(-1) = (-1)^3 - 2(-1)^2 + 7(-1) + 10$

$= -1 - 2(1) - 7 + 10 = -1 - 2 - 7 + 10 = 0$

Because $f(-1) = 0 \Rightarrow -1$ is a root. Therefore, $(x + 1)$ is a factor.

2 (b) (ii)

STEPS

1. Guess at a root (unless a root is given) by substituting in numbers $0, 1, -1, 2, -2, \ldots$ until you get zero.
2. Using the factor theorem, form a factor from the root.
3. Divide the cubic by the factor to get the quadratic.
4. Solve the quadratic by factorising or using formula **2**.
5. Write down the three roots.

Steps 1 and 2 are already done in 2 (b) (i).

3. Divide the factor into the cubic.

$\therefore 2x^3 + 7x^2 + x - 10 = (x + 2)(2x^2 + 3x - 5) = 0$

$$\begin{array}{r|l} & 2x^2 + 3x - 5 \\ \hline x + 2 & 2x^3 + 7x^2 + x - 10 \\ & \mp 2x^3 \mp 4x^2 \\ \hline & \quad 3x^2 + x - 10 \\ & \quad \mp 3x^2 \mp 6x \\ \hline & \qquad -5x - 10 \\ & \qquad \pm 5x \pm 10 \\ \hline & \qquad\quad 0 \end{array}$$

4. $2x^2 + 3x - 5 = (2x + 5)(x - 1)$

5. $\therefore 2x^3 + 7x^2 + x - 10 = (x + 2)(2x + 5)(x - 1) = 0$

Set each factor equal to zero and solve for x.

$\therefore x = -2, -\frac{5}{2}, 1$

2000

3 (b) (i) Show that $x = 2$ is a root of $3x^3 + 8x^2 - 33x + 10 = 0$.

(ii) Find the other roots of $3x^3 + 8x^2 - 33x + 10 = 0$.

SOLUTION

3 (b) (i)

$f(x) = 3x^3 + 8x^2 - 33x + 10 \Rightarrow f(2) = 3(2)^3 + 8(2)^2 - 33(2) + 10$

$= 24 + 32 - 66 + 10 = 0$

3 (b) (ii)

STEPS

1. Guess at a root (unless a root is given) by substituting in numbers $0, 1, -1, 2, -2, \ldots$ until you get zero.
2. Using the factor theorem, form a factor from the root.
3. Divide the cubic by the factor to get the quadratic.
4. Solve the quadratic by factorising or using formula **2**.
5. Write down the three roots.

1. $f(2) = 0$

2. $\therefore (x-2)$ is a factor.

3. Divide this factor into the cubic as shown to the right.

$\therefore 3x^3 + 8x^2 - 33x + 10 = (x-2)(3x^2 + 14x - 5) = 0$

$$\begin{array}{r|l} & 3x^2 + 14x - 5 \\ \hline x-2 & 3x^3 + 8x^2 - 33x + 10 \\ & \mp 3x^3 \pm 6x^2 \\ \hline & 14x^2 - 33x + 10 \\ & \mp 14x^2 \pm 28x \\ \hline & -5x + 10 \\ & \pm 5x \mp 10 \\ \hline & 0 \end{array}$$

4. Factorise the quadratic.

$3x^2 + 14x - 5 = (3x-1)(x+5)$

5. $\therefore 3x^3 + 8x^2 - 33x + 10 = (x-2)(3x-1)(x+5) = 0$

Set each factor equal to zero and solve for x.

$\therefore x = -5, \frac{1}{3}, 2$

1999

3 (c) Show that $6x^2+5x-4$ is a factor of $6x^3+17x^2+6x-8$.

Hence, or otherwise, find the roots of $6x^3+17x^2+6x-8=0$.

SOLUTION

To show $6x^2+5x-4$ is a factor of $6x^3+17x^2+6x-8$, divide it in and show the remainder is zero.

$$\begin{array}{r|l} & \quad x+2 \\ 6x^2+5x-4 & \; 6x^3+17x^2+6x-8 \\ & \mp 6x^3 \mp 5x^2 \pm 4x \\ & \qquad 12x^2+10x-8 \\ & \qquad \mp 12x^2 \mp 10x \pm 8 \\ & \qquad\qquad 0 \end{array}$$

$\therefore 6x^3+17x^2+6x-8=(6x^2+5x-4)(x+2)=0$

Factorise the quadratic: $6x^2+5x-4=(3x+4)(2x-1)$

$\therefore 6x^3+17x^2+6x-8=(3x+4)(2x-1)(x+2)=0$

Set each factor is equal to zero and solve for x.

$\therefore x=-\frac{4}{3}, \frac{1}{2}, -2$

1998

3 (b) (i) If $(x-2)$ is a factor of $3x^3+x^2+kx+6$, find the value of k.

(ii) Write down an equation which has three roots of value $-3, 1$ and 5.

SOLUTION

3 (b) (i) The factor theorem states that:

> If $(x-k)$ is a factor of $f(x)$ then k is a root of $f(x)=0$,
> i.e. $f(k)=0$ and vice versa.

If $(x-2)$ is a factor of $f(x)=3x^3+x^2+kx+6 \Rightarrow f(2)=0$.

$\therefore f(2)=3(2)^3+(2)^2+k(2)+6=0$

$\Rightarrow 24+4+2k+6=0 \Rightarrow 2k+34=0$

$\Rightarrow 2k=-34 \Rightarrow k=-17$

CONT...

3 (b) (ii)

-3 is a root $\Rightarrow (x+3)$ is a factor.

1 is a root $\Rightarrow (x-1)$ is a factor.

5 is a root $\Rightarrow (x-5)$ is a factor.

Cubic equation: $\Rightarrow (x+3)(x-1)(x-5)=0$

$\Rightarrow (x+3)(x^2-6x+5)=0$

$\Rightarrow x^3-6x^2+5x+3x^2-18x+15=0$

$\Rightarrow x^3-3x^2-13x+15=0$

1997

3 (b) Solve the equation

$$2x^3+3x^2-5x-6=0.$$

SOLUTION

Solving cubic equations:

STEPS

1. Guess at a root (unless a root is given) by substituting in numbers $0, 1, -1, 2, -2, ...$ until you get zero.
2. Using the factor theorem, form a factor from the root.
3. Divide the cubic by the factor to get the quadratic.
4. Solve the quadratic by factorising or using formula **2**.
5. Write down the three roots.

1. $f(1)=2(1)^3+3(1)^2-5(1)-6=2+3-5-6=-6\neq 0$

$f(-1)=2(-1)^3+3(-1)^2-5(-1)-6=-2+3+5-6=0$

2. $\therefore (x+1)$ is a factor.

3. Divide this factor into the cubic as shown on the right.

$\therefore 2x^3+3x^2-5x-6=(x+1)(2x^2+x-6)=0$

$$\begin{array}{r|l} & 2x^2+x-6 \\ \hline x+1 & 2x^3+3x^2-5x-6 \\ & \mp 2x^3 \mp 2x^2 \\ \hline & x^2-5x-6 \\ & \mp x^2 \mp x \\ \hline & -6x-6 \\ & \pm 6x \pm 6 \\ \hline & 0 \end{array}$$

4. Factorise the quadratic.

$2x^2+x-6=(2x-3)(x+2)$

5. $\therefore 2x^3+3x^2-5x-6=(x+1)(2x-3)(x+2)=0$

Set each factor equal to zero and solve for x.

$\therefore x=-2, -1, \frac{3}{2}$

1996

3 (b) Find the roots of the equation

$$2x^3 - 5x^2 + x + 2 = 0.$$

SOLUTION

STEPS

1. Guess at a root (unless a root is given) by substituting in numbers $0, 1, -1, 2, -2, \ldots$ until you get zero.
2. Using the factor theorem, form a factor from the root.
3. Divide the cubic by the factor to get the quadratic.
4. Solve the quadratic by factorising or using formula **2**.
5. Write down the three roots.

1. $f(x) = 2x^3 - 5x^2 + x + 2 \Rightarrow f(1) = 2(1)^3 - 5(1)^2 + (1) + 2$

$\Rightarrow f(1) = 2 - 5 + 1 + 2 = 0$

2. $\therefore (x-1)$ is a factor.

3. Divide this factor into the cubic as shown to the right.

$\therefore 2x^3 - 5x^2 + x + 2 = (x-1)(2x^2 - 3x - 2) = 0$

4. Factorise the quadratic.

$2x^2 - 3x - 2 = (2x+1)(x-2)$

5. $\therefore 2x^3 - 5x^2 + x + 2 = (x-1)(2x+1)(x-2) = 0$

Set each factor equal to zero and solve for x.

$\therefore x = -\frac{1}{2}, 1, 2$

$$\begin{array}{r|l} & 2x^2 - 3x - 2 \\ \hline x - 1 & 2x^3 - 5x^2 + x + 2 \\ & \mp 2x^3 \pm 2x^2 \\ \hline & \quad -3x^2 + x + 2 \\ & \quad \mp 3x^2 \mp 3x \\ \hline & \qquad -2x + 2 \\ & \qquad \pm 2x \mp 2 \\ \hline & \qquad\quad 0 \end{array}$$