ALGEBRA (Q 2 & 3, PAPER 1)

LESSON NO. 3: QUADRATIC EQUATIONS

2007

2 (c) (i) Solve the equation $x - \frac{1}{x} = 2$ and write your solutions in the form $a \pm \sqrt{b}$, where $a, b \in \mathbf{N}$.

(ii) Verify **one** of your solutions.

SOLUTION

2 (c) (i)

$x - \frac{1}{x} = 2$ [Multiply across by x.]

$x^2 - 1 = 2x \Rightarrow x^2 - 2x - 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{.......} \; \textbf{2}$$

$$\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2}$$

$a = 1$
$b = -2$
$c = -1$

$$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Note: $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$

$\therefore x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$.

2 (c) (ii)

Verify $x = 1 - \sqrt{2}$ is a solution by sustituting it back into the original equation.

$$x - \tfrac{1}{x} = 2 \Rightarrow (1+\sqrt{2}) - \frac{1}{(1+\sqrt{2})} = 2 \text{ [This needs to be proved]}$$

$$(1+\sqrt{2}) - \frac{1}{(1+\sqrt{2})} \times \frac{(1-\sqrt{2})}{(1-\sqrt{2})}$$ [Multiply above and below by the conjugate of the denominator.]

$$= (1+\sqrt{2}) - \frac{1-\sqrt{2}}{1-\sqrt{2}+\sqrt{2}-2} = (1+\sqrt{2}) - \frac{1-\sqrt{2}}{-1}$$

$$= 1 + \sqrt{2} + 1 - \sqrt{2} = 2$$

2006

3 (c) Solve for x

$$x = \frac{3+2x}{x-2}, \; x \neq 2$$

and give your solutions in the form form $a \pm \sqrt{b}$, where $a, b \in \mathbf{N}$.
Write one of your solutions correct to two decimal places. Using this value, show that the difference between the values of the left hand side and the right hand side of the given equation is less that 0.1.

SOLUTION

$x = \frac{3+2x}{x-2} \Rightarrow x(x-2) = 3+2x$ [Multiply both sides by $(x-2)$.]

$\Rightarrow x^2 - 2x = 3 + 2x$

$\Rightarrow x^2 - 4x - 3 = 0$ [Solve using formula **2**.]

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \; \ldots\ldots\ldots \; \mathbf{2}$$

$a = 1$
$b = -4$
$c = -3$

$\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)} = \frac{4 \pm \sqrt{16+12}}{2} = \frac{4 \pm \sqrt{28}}{2}$

$= \frac{4 \pm 2\sqrt{7}}{2} = 2 \pm \sqrt{7}$

Note: $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

Write $2+\sqrt{7}$ in decimal form: $2+\sqrt{7} = 4.65$

LHS: $x = 4.65$

RHS: $\frac{3+2(4.65)}{4.65-2} = 4.64$ [Using your calculator.]

Subtracting both sides: $4.65 - 4.64 = 0.01 < 0.1$

2004

2 (b) (i) Solve $2x^2 - 7x + 3 = 0$.

(ii) Show that $x-2$ is a factor of $x^3 - 3x^2 - x + 6$.

SOLUTION

2 (b) (i)

$2x^2 - 7x + 3 = 0$

$\Rightarrow (2x-1)(x-3) = 0$ [Set each factor equal to zero and solve for x.]

$\therefore (2x-1) = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

$\therefore (x-3) = 0 \Rightarrow x = 3$

CONT....

2 (b) (ii)

The factor theorem states that:

If $(x-k)$ is a factor of $f(x)$ then k is a root of $f(x)=0$, i.e. $f(k)=0$ and vice versa.

If $x-2$ is a factor $\Rightarrow 2$ is a root. Substitute 2 in for x.

$f(x)=x^3-3x^2-x+6 \Rightarrow f(2)=(2)^3-3(2)^2-(2)+6$

$=8-12-2+6=0$

Because $f(2)=0 \Rightarrow 2$ is a root. Therefore, $x-2$ is a factor.

2003

2 (c) Solve for x the equation

$$\frac{3}{x+1}+\frac{1}{x-1}=1.$$

Give your answers in the form $a \pm \sqrt{b}$, where $a, b \in \mathbf{N}$.

3 (c) (i) Simplify $(x+\sqrt{a-x})(x-\sqrt{a-x})$, where $a-x \geq 0$.

(ii) Given that $x = 3$ is a solution of the equation $(x+\sqrt{a-x})(x-\sqrt{a-x})=0$, find the value of a.

(iii) Hence, find the other solution of the equation in part **(ii)**, and verify your answer.

SOLUTION

2 (c)

$\frac{3}{x+1}+\frac{1}{x-1}=1$ [Multiply each term by $(x+1)(x-1)$.]

$\Rightarrow \frac{3(x+1)(x-1)}{(x+1)}+\frac{1(x+1)(x-1)}{(x-1)}=1(x+1)(x-1)$

$\Rightarrow 3(x-1)+(x+1)=(x+1)(x-1)$ [Multiply out the brackets.]

$\Rightarrow 3x-3+x+1=x^2-1$

$\Rightarrow x^2-4x+1=0$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ **2**

$a=1$
$b=-4$
$c=1$

$\therefore x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}$

$=\frac{4\pm 2\sqrt{3}}{2}=2\pm\sqrt{3}$

Note: $\sqrt{12}=\sqrt{4\times 3}=2\sqrt{3}$

CONT....

3 (c) (i)

$(x+\sqrt{a-x})(x-\sqrt{a-x}) = x^2 - x\sqrt{a-x} + x\sqrt{a-x} - \sqrt{a-x}\sqrt{a-x}$

$= x^2 - (a-x) = x^2 + x - a$

3 (c) (ii)

$(x+\sqrt{a-x})(x-\sqrt{a-x}) = 0 \Rightarrow x^2 + x - a = 0$

$x = 3 \Rightarrow 3^2 + 3 - a = 0 \Rightarrow 9 + 3 - a = 0 \Rightarrow a = 12$

3 (c) (iii)

$x^2 + x - 12 = 0 \Rightarrow (x+4)(x-3) = 0$

$\therefore x = -4, 3$

Therefore, the other solution is $x = -4$.

2001

3 (b) (i) Simplify $(x+\sqrt{x})(x-\sqrt{x})$ when $x > 0$.

(ii) Hence, or otherwise, find the value of x for which $(x+\sqrt{x})(x-\sqrt{x}) = 6$.

Solution

3 (b) (i)

$(x+\sqrt{x})(x-\sqrt{x}) = x^2 - x\sqrt{x} + x\sqrt{x} - \sqrt{x}\sqrt{x}$

$= x^2 - x$

3 (b) (ii)

$(x+\sqrt{x})(x-\sqrt{x}) = 6 \Rightarrow x^2 - x = 6$

$\Rightarrow x^2 - x - 6 = 0$ [Factorise the quadratic.]

$\Rightarrow (x-3)(x+2) = 0$ [Set each factor equal to zero and solve for x.]

$\therefore x = -2, 3$

1999

2 (c) Solve for x

$$\frac{3}{2x-1}=1+\frac{2x}{x+2},\ x\neq\tfrac{1}{2}\text{ and } x\neq -2.$$

SOLUTION

$\frac{3}{2x-1}=1+\frac{2x}{x+2}$ [Multiply each term by $(2x-1)(x+2)$.]

$\Rightarrow \frac{3(2x-1)(x+2)}{(2x-1)}=1(2x-1)(x+2)+\frac{2x(2x-1)(x+2)}{(x+2)}$ [Cancel brackets that are the same.]

$\Rightarrow 3(x+2)=1(2x-1)(x+2)+2x(2x-1)$ [Multiply out the brackets.]

$\Rightarrow 3x+6=2x^2+3x-2+4x^2-2x$ [Bring all the terms to the right.]

$\Rightarrow 0=6x^2-2x-8$ [Divide across by 2.]

$\Rightarrow 3x^2-x-4=0$ [Factorise the quadratic.]

$\Rightarrow (3x-4)(x+1)=0$ [Set eqch factor equal to zero and solve for x.]

$\therefore x=-1,\ \tfrac{4}{3}$

1998

3 (c) (i) Write $\frac{1}{x+1}+\frac{2}{x-3}$ as a single fraction where $x\neq -1$ and $x\neq 3$.

(ii) Hence, or otherwise, find, correct to one place of decimals, the two solutions of

$$\frac{1}{x+1}+\frac{2}{x-3}=1,\ x\neq -1,\ x\neq 3.$$

SOLUTION

3 (c) (i)

$\frac{1}{x+1}+\frac{2}{x-3}$ [Get the common denominator which is $(x+1)(x-3)$.]

$=\frac{1(x-3)+2(x+1)}{(x+1)(x-3)}$ [Multiply out the top and tidy up.]

$=\frac{x-3+2x+2}{(x+1)(x-3)}=\frac{3x-1}{(x+1)(x-3)}$

CONT....

3 (c) (ii)

$\frac{1}{x+1}+\frac{2}{x-3}=1 \Rightarrow \frac{3x-1}{(x+1)(x-3)}=1$ [Multiply both sides by $(x+1)(x-3)$.]

$\Rightarrow 3x-1=1(x+1)(x-3)$ [Multiply out the brackets.]

$\Rightarrow 3x-1=x^2-2x-3$ [Bring all terms to one side.]

$\Rightarrow 0=x^2-5x-2$

$\therefore x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-2)}}{2(1)}=\frac{5\pm\sqrt{25+8}}{2}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ 2

$a=1$
$b=-5$
$c=-2$

$=\frac{5\pm\sqrt{33}}{2}$

$\therefore x=-0.4,\ 5.4$ [Using calculator.]

1997

2 (c) Simplify

$$\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right) \text{ where } x>0.$$

Hence solve for x

$$\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)=8 \text{ where } x>0.$$

SOLUTION

$\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)$ [Multiply each term in the first bracket by each term in the second bracket.]

$=\sqrt{x}\sqrt{x}-\sqrt{x}\frac{3}{\sqrt{x}}-\frac{3}{\sqrt{x}}\sqrt{x}-\frac{3}{\sqrt{x}}\times\frac{3}{\sqrt{x}}$

$(\sqrt{x})^2=\sqrt{x}\times\sqrt{x}=x$

$=x-\frac{9}{x}$

$\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)=8 \Rightarrow x-\frac{9}{x}=8$ [Multiply each term by x.]

$\Rightarrow x^2-9=8x$

$\Rightarrow x^2-8x-9=0$ [Factorise the quadratic.]

$\Rightarrow (x-9)(x+1)=0$

$\therefore x=-1,\ 9$

1996

2 (c) Solve

$$\frac{x-1}{x}-\frac{3x}{x-1}=2,\ x\neq 0 \text{ and } x\neq 1.$$

SOLUTION

$\frac{x-1}{x}-\frac{3x}{x-1}=2$ [Multiply each term by $x(x-1)$.]

$\frac{(x-1)x(x-1)}{x}-\frac{3x(x)(x-1)}{(x-1)}=2x(x-1)$ [Cancel brackets that are the same.]

$\Rightarrow (x-1)(x-1)-3x^2=2x^2-2x$ [Multiply out the brackets.]

$\Rightarrow x^2-2x+1-3x^2=2x^2-2x$ [Bring all terms to the left hand side.]

$\Rightarrow -4x^2+1=0$ [Multiply across by -1.]

$\Rightarrow 4x^2-1=0$ [Factorise the quadratic.]

$\Rightarrow (2x+1)(2x-1)=0$ $a^2-b^2=(a+b)(a-b)$ 1

$\therefore x=-\frac{1}{2},\ \frac{1}{2}$