7. **(a)** Find the slope of the tangent to the curve $x^2 + y^3 = x - 2$ at the point (3, –2).

(b) A curve is defined by the parametric equations

$$x = \frac{t-1}{t+1} \text{ and } y = \frac{-4t}{(t+1)^2}, \text{ where } t \neq -1.$$

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

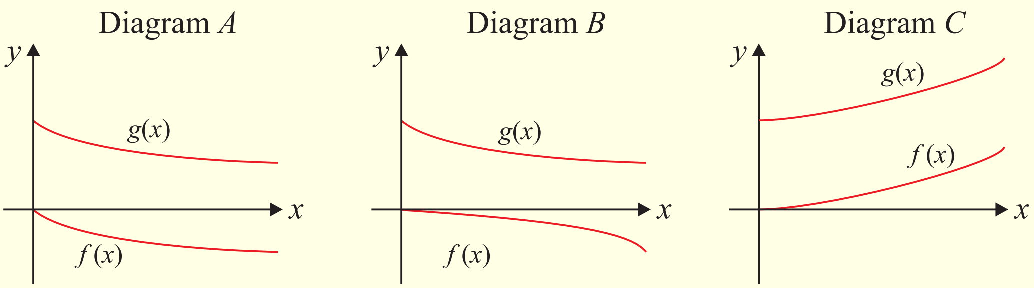
(ii) Hence find $\frac{dy}{dx}$, and express your answer in terms of x.

(c) The functions f and g are defined on the domain $x \in \mathbb{R} \setminus \{-1, 0\}$ as follows:

$$f : x \to \tan^{-1}\left(\frac{-x}{x+1}\right) \text{ and } g : x \to \tan^{-1}\left(\frac{x+1}{x}\right).$$

(i) Show that $f'(x) = \frac{-1}{2x^2 + 2x + 1}$.

(ii) It can be shown that $f'(x) = g'(x)$.
One of the three diagrams A, B, or C below represents parts of the graphs of f and g. Based only on the derivatives, state which diagram is the correct one, and state also why each of the other two diagrams is incorrect.



Solution

7 (a)

$$2x + 3y^2 \times \frac{dy}{dx} = 1$$

$$3y^2 \times \frac{dy}{dx} = 1 - 2x$$

$$\frac{dy}{dx} = \frac{1-2x}{3y^2}$$

$$\left(\frac{dy}{dx}\right)_{(3,\,-2)} = \frac{1-2(3)}{3(-2)^2} = \frac{1-6}{3(4)} = -\frac{5}{12}$$

7 (b) (i)

Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$x = \frac{t-1}{t+1}$$

$$\frac{dx}{dt} = \frac{(t+1)(1)-(t-1)(1)}{(t+1)^2}$$

$$= \frac{t+1-t+1}{(t+1)^2}$$

$$= \frac{2}{(t+1)^2}$$

$$y = \frac{-4t}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{(t+1)^2(-4)-(-4t)2(t+1)(1)}{(t+1)^4}$$

$$= \frac{-4(t^2+2t+1)+8t(t+1)}{(t+1)^4}$$

$$= \frac{-4t^2-8t-4+8t^2+8t}{(t+1)^4}$$

$$= \frac{4t^2-4}{(t+1)^4}$$

$$= \frac{4(t^2-1)}{(t+1)^4} = \frac{4(t-1)(t+1)}{(t+1)^4}$$

$$= \frac{4(t-1)}{(t+1)^3}$$

7 (b) (ii)

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{4(t-1)}{(t+1)^3}}{\frac{2}{(t+1)^2}} = \frac{4(t-1)}{(t+1)^3} \times \frac{(t+1)^2}{2} = \frac{2(t-1)}{(t+1)} = 2x$$

7 (c) (i)

$$f(x)=\tan^{-1}\left(\frac{-x}{x+1}\right)$$

$$f'(x)=\frac{1}{1+\left(\frac{-x}{x+1}\right)^2}\times\frac{(x+1)(-1)-(-x)(1)}{(x+1)^2}$$

$$y=\tan^{-1}f(x)\Rightarrow\frac{dy}{dx}=\frac{1}{1+f(x)^2}\times f'(x)$$

$$=\frac{1}{1+\frac{x^2}{(x+1)^2}}\times\frac{-x-1+x}{(x+1)^2}$$

$$=\frac{(x+1)^2}{(x+1)^2+x^2}\times\frac{-1}{(x+1)^2}$$

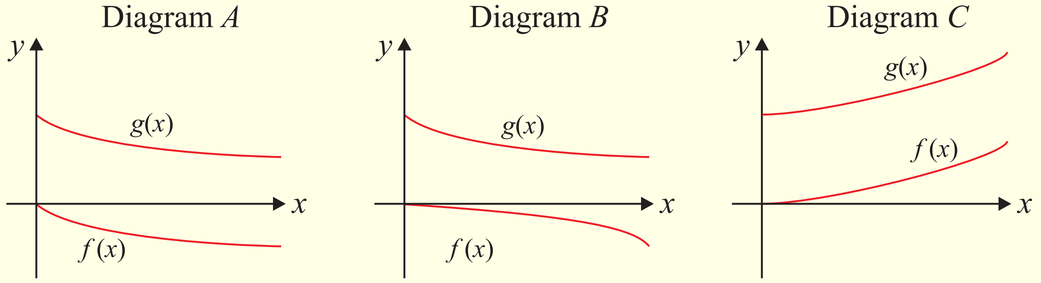
$$=\frac{-1}{x^2+2x+1+x^2}$$

$$=\frac{-1}{2x^2+2x+1}$$

7 (c) (ii)

$$f'(x)=g'(x)=\frac{-1}{2x^2+2x+1}=\frac{-1}{x^2+2x+1+x^2}=\frac{-1}{(x+1)^2+x^2}<0 \text{ for all } x.$$

Therefore, the graph for $f(x)$ is always decreasing.
$g(x)$ has the same slope and is also decreasing.



A is correct: Both functions are decreasing with the same slopes everywhere.
B is incorrect: Both slopes are not the same everywhere.
C is incorrect: Both functions are increasing.