T2 (MEDIUM)

Example 39: Evaluate $\int_0^2 \frac{dx}{\sqrt{16-4x^2}}$.

SOLUTION

You must get a 1 in front of x^2. So take out 4 on the bottom as a factor. When it comes through the square root sign it becomes 2.

$$I = \int_0^2 \frac{dx}{\sqrt{16-4x^2}} = \tfrac{1}{2}\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \tfrac{1}{2}\int_0^2 \frac{dx}{\sqrt{(2)^2-(1x)^2}}$$

$$= \tfrac{1}{2}[\sin^{-1} \tfrac{x}{2}]_0^2 = \tfrac{1}{2}(\sin^{-1}1 - \sin^{-1}0) = \tfrac{1}{2}(\tfrac{\pi}{2} - 0) = \tfrac{\pi}{4}$$

T3 (HARD)

Example 40: Evaluate $\int_4^7 \frac{dx}{\sqrt{8x-x^2-7}}$.

SOLUTION

We have to put $8x - x^2 - 7$ into the form $a^2 - (1x \pm b)^2$ by completing the square (CS). This is difficult because of the $-x^2$.
Trick: Take out a factor of -1 before completing the square.

$$I = \int_4^7 \frac{dx}{\sqrt{8x-x^2-7}}$$

$$= \int_4^7 \frac{dx}{\sqrt{(3)^2-(1x-4)^2}}$$

CS: $8x - x^2 - 7$
$= -\{(x^2 - 8x) + 7\}$
$= -\{(x - 4)^2 - 16 + 7\}$
$= -\{(x - 4)^2 - 9\}$
$= 9 - (x - 4)^2$

$$= \left[\sin^{-1}\left(\frac{x-4}{3}\right)\right]_4^7 = \sin^{-1}1 - \sin^{-1}0 = \tfrac{\pi}{2}$$

[C] **The Louser: The square root of the quadratic function**

$$\int \sqrt{a^2 - 1x^2}\, dx$$

TRICK

You must make a special substitution. Let $x = a\sin u$

Example 41: Evaluate $\int_0^{\frac{3}{2}} \sqrt{9 - x^2}\, dx$.

SOLUTION

$$\text{I} = \int_0^{\frac{3}{2}} \sqrt{9 - x^2}\, dx = \int_0^{\frac{3}{2}} \sqrt{(3)^2 - (x)^2}\, dx$$

Let $x = 3\sin u \Rightarrow dx = 3\cos u\, du$

Changing the limits

$x = 3\sin u$

$x = 0 \Rightarrow u = 0$

$x = \frac{3}{2} \Rightarrow u = \frac{\pi}{6}$

$$\therefore \text{I} = 3\int_0^{\frac{\pi}{6}} \sqrt{9 - 9\sin^2 u}\, \cos u\, du$$

$$= 9\int_0^{\frac{\pi}{6}} \sqrt{1 - \sin^2 u}\, \cos u\, du$$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$$= 9\int_0^{\frac{\pi}{6}} \cos^2 u\, du = \frac{9}{2}\int_0^{\frac{\pi}{6}} (1 + \cos 2u)\, du = \frac{9}{2}[u + \tfrac{1}{2}\sin 2u]_0^{\frac{\pi}{6}}$$

$$= \frac{9}{2}\{(\tfrac{\pi}{6} + \tfrac{1}{2}\sin(\tfrac{\pi}{3})) - (0 + 0)\} = \frac{9}{2}\{\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{4}\}$$

Example 42: Evaluate $\int_0^1 \sqrt{16 - 4x^2}\, dx$.

SOLUTION

Changing the limits

$x = 2\sin u$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = \frac{\pi}{6}$

$$\text{I} = \int_0^1 \sqrt{16 - 4x^2}\, dx = 2\int_0^1 \sqrt{4 - x^2}\, dx$$

Let $x = 2\sin u \Rightarrow dx = 2\cos u\, du$

$$\text{I} = 4\int_0^{\frac{\pi}{6}} \sqrt{4 - 4\sin^2 u}\, \cos u\, du = 8\int_0^{\frac{\pi}{6}} \cos^2 u\, du = 4\int_0^{\frac{\pi}{6}} (1 + \cos 2u)$$

$$= 4[u + \tfrac{\sin 2u}{2}]_0^{\frac{\pi}{6}} = 4(\tfrac{\pi}{6} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}) = \tfrac{2\pi}{3} + \sqrt{3}$$