Sean Burke's Questions and Solutions

Question 1

If $p, q, r, s \geq 0$ prove that:

(i) $(p+q)(q+r)(r+p) \geq 8pqr$

(ii) $(pq+rs)(pr+qs) \geq 4pqrs$

Solution

(i) $(p+q)(q+r)(r+p) \geq 8pqr \Rightarrow (p+q)(q+r)(r+p) - 8pqr \geq 0$ [This is what you prove]

$(p+q)(q+r)(r+p) - 8pqr$

$= (pq + pr + q^2 + qr)(r+p) - 8pqr$

$= pqr + p^2q + pr^2 + p^2r + q^2r + q^2p + qr^2 + pqr - 8pqr$

$= p^2q + pr^2 + p^2r + q^2r + q^2p + qr^2 - 6pqr$

$= p^2q - 2pqr + qr^2 + q^2p - 2pqr + pr^2 + p^2r - 2pqr + rq^2$

$= (pq - qr)(p-r) + (qp - pr)(q-r) + (pr - rq)(p-q)$

$= q(p-r)(p-r) + p(q-r)(q-r) + r(p-q)(p-q)$

$= q(p-r)^2 + p(q-r)^2 + r(p-q)^2 \geq 0$

(ii) $(pq+rs)(pr+qs) \geq 4pqrs \Rightarrow (pq+rs)(pr+qs) - 4pqrs \geq 0$

$(pq+rs)(pr+qs) - 4pqrs$

$= p^2qr + psq^2 + r^2ps + rqs^2 - 4pqrs$

$= p^2qr - 2pqrs + rqs^2 + r^2ps - 2pqrs + psq^2$

$= (pq - qs)(pr - rs) + (rp - pq)(rs - sq)$

$= qr(p-s)(p-s) + ps(r-q)(r-q)$

$= qr(p-s)^2 + ps(r-q)^2 \geq 0$

Question 2

(i) Prove that if $(x-k)$ is a factor of $f(x)=ax^3+bx^2+cx+d$, then $f(k)=0$.

(ii) If a, b, c, d are integers, deduce that k is a factor of d.

PROOF OF FACTOR THEOREM

$f(x)=ax^3+bx^2+cx+d$

$f(k)=ak^3+bk^2+ck+d$

$\therefore f(x)-f(k)=a(x^3-k^3)+b(x^2-k^2)+c(x-k)$

$=(x-k)\{ax^2+akx+ak^2+bx+bk+c\}=(x-k)g(x)$

$\therefore f(x)=f(k)+(x-k)g(x)$

(i) $f(k)=0 \Rightarrow f(x)=(x-k)g(x) \therefore x-k$ is a factor.

(ii) $x-k$ is a factor $\Rightarrow f(k)=0$.

(ii) As $(x-k)$ is a factor of $f(x)=ax^3+bx^2+cx+d$, when you divide $(x-k)$ into the cubic function the remainder has to be zero.

$$\begin{array}{r|l}
 & ax^2+(ak+b)x+[c+k(ak+b)] \\
\hline
x-k & ax^3+bx^2+cx+d \\
 & \mp ax^3 \pm akx^2 \\
\hline
 & (ak+b)x^2+cx+d \\
 & \mp(ak+b)x^2 \pm k(ak+b)x \\
\hline
 & [c+k(ak+b)]x+d \\
 & \mp[c+k(ak+b)]x \pm k[c+k(ak+b)] \\
\hline
 & d+k[c+k(ak+b)]
\end{array}$$

As the remainder is zero $\Rightarrow d+k[c+k(ak+b)]=0 \Rightarrow d=-k[c+k(ak+b)]$

As you can see, d equals k times the square bracket, i.e. k is a factor of d.

This information is useful when solving cubic equations. For example, say you were asked to find the roots of the cubic equation, $12x^3-11x^2-2x+1=0$, given that there exists an integer root. By trial and error, you substitute in various numbers to see what number works. However, you can speed up the process because you know that the number you substitute in, k, is a factor of 1 (d). So the only numbers that could work are +1 or -1.

Question 3

(i) If $f(x) = ax^3 + (a+b)x^2 + (a+2b)x + 1$ is exactly divisible by $(x+1)$ express b in terms of a,

(ii) find the quotient when $f(x)$ is divided by $(x-1)$ expressing the coefficients in terms of a only.

$$\begin{array}{r|l} & ax^2 + bx + (a+b) \\ \hline x+1 & ax^3 + (a+b)x^2 + (a+2b)x + 1 \\ & \mp ax^3 \mp ax^2 \\ \hline & \quad bx^2 + (a+2b)x + 1 \\ & \quad \mp bx^2 \mp bx \\ \hline & \qquad (a+b)x + 1 \\ & \qquad \mp(a+b)x \mp (a+b) \\ \hline & \qquad\qquad 1-(a+b) \end{array}$$

The remainder is zero $1-(a+b) = 0 \Rightarrow 1-a-b = 0 \Rightarrow b = 1-a$

$\Rightarrow f(x) = ax^3 + x^2 + (2-a)x + 1$

$$\begin{array}{r|l} & ax^2 + (1+a)x + 3 \\ \hline x-1 & ax^3 + x^2 + (2-a)x + 1 \\ & \mp ax^3 \pm ax^2 \\ \hline & \quad (1+a)x^2 + (2-a)x + 1 \\ & \quad \mp(1+a)x^2 \pm (1+a)x \\ \hline & \qquad 3x + 1 \\ & \qquad \mp 3x \pm 3 \\ \hline & \qquad\qquad 4 \end{array}$$

Question 4

Evaluate $\dfrac{9^{n+2} - 2\times 3^{2n+3}}{3^{2n+1} - 9^n}$.

Solution

$$\frac{9^{n+2} - 2\times 3^{2n+3}}{3^{2n+1} - 9^n} = \frac{(3^2)^{n+2} - 2\times 3^{2n+3}}{3^{2n+1} - (3^2)^n} = \frac{3^{2n+4} - 2\times 3^{2n+3}}{3^{2n+1} - 3^{2n}}$$

$$= \frac{3^{2n+3}[3^1 - 2]}{3^{2n}[3^1 - 1]} = \frac{3^3(1)}{2} = \frac{27}{2}$$